Gauge Invariant Cosmological Perturbation Theory

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December 18, 2020

Motivation

Where do inhomogeneities in the universe come from?

How to construct a theory of gauge invariant cosmological perturbations?

What are the physical degrees of freedom that have an imprint in the CMB?

Outline

1. Electrodynamics
2. Gravity on Flat Space
3. Gravity on FRW Backgrounds

Electrodynamics

Maxwell's Theory:

$$S = -\frac{1}{4}\int \mathrm{d}^4x\, F^{\mu\nu}F_{\mu\nu}\,, \quad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}\,, \text{ invariant under } \delta A_\mu = \partial_\mu \Lambda\,.$$

Goal: Write Maxwell's theory in manifestly gauge invariant form.

Decomposition of vector field:

$$A_i = \widehat{A}_i + \partial_i\psi\,, \qquad \partial_i \widehat{A}^i = 0\,,$$
$$\psi = \Delta^{-1}(\partial_i A^i)\,, \quad \Delta \equiv \partial^i\partial_i \text{ invertible.}$$

Components of A_μ transform as: $\delta A_0 = \dot{\Lambda}\,, \quad \delta\widehat{A}_i = 0\,, \quad \delta\psi = \Lambda\,.$
Maxwell's equations $\partial_\mu F^{\mu\nu} = 0$ in terms of $\widehat{A}_i$ and $\Phi \equiv A_0 - \dot{\psi}$:

$$\Delta\Phi = 0 \quad \rightarrow \quad \Phi = 0\,, \quad \text{(using } \Delta \text{ invertible)}$$
$$\Box\,\widehat{A}_i + \partial_i\dot{\Phi} = 0 \quad \rightarrow \quad \Box\widehat{A}_i = 0\,.$$

Manifestly gauge-invariant action: $\quad S = \frac{1}{2}\int \mathrm{d}^4x\big(\widehat{A}^i\,\Box\,\widehat{A}_i - \Phi\Delta\Phi\big)\,.$

Gravity on Flat Space

Einstein-Hilbert Action:

$$S = \int d^4x \sqrt{-g}\, R\,,$$

invariant under $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \equiv \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}$.
Expanding around flat space, $g_{\mu\nu}(t,x) = \eta_{\mu\nu} + h_{\mu\nu}(t,x)$:

$$S = -\frac{1}{2}\int d^4x\, h^{\mu\nu} G_{\mu\nu}(h)\,, \quad \text{invariant under } \delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu\,.$$

Decompose $h_{\mu\nu}$:

$$\begin{aligned} h_{00} &= -2\phi\,, \\ h_{0i} &= B_i + \partial_i B\,, \\ h_{ij} &= \widehat{h}_{ij} + \partial_i E_j + \partial_j E_i + \partial_i \partial_j E + \tfrac{1}{3}\delta_{ij}\big(C - \Delta E\big)\,, \end{aligned}$$

where $\quad \partial^i B_i = \partial^i E_i = 0\,, \quad \partial^i \widehat{h}_{ij} = \delta^{ij}\widehat{h}_{ij} = 0\,.$

Gravity on Flat Space

Decompose gauge parameter (ξ_0, ξ_i): $\xi_i = \zeta_i + \partial_i \chi$ with $\partial^i \zeta_i = 0$.

$$
\begin{aligned}
\delta h_{00} &= -2\delta\phi = 2\dot{\xi}_0 \,, \\
\delta h_{0i} &= \delta(B_i + \partial_i B) = \dot{\zeta}_i + \partial_i(\dot{\chi} + \xi_0) \,, \\
\delta h_{ij} &= \delta\left[\widehat{h}_{ij} + \partial_i E_j + \partial_j E_i + \partial_i \partial_j E + \tfrac{1}{3}\delta_{ij}(C - \Delta E)\right] \\
&= \partial_i \zeta_j + \partial_j \zeta_i + 2\partial_i \partial_j \chi
\end{aligned}
$$

Gauge transformations of SVT components:

$$
\begin{aligned}
&\delta\phi = -2\dot{\xi}_0 \,, \\
&\delta B_i = \dot{\zeta}_i \,, \quad \delta B = \dot{\chi} + \xi_0 \,, \\
&\delta E_i = \zeta_i \,, \quad \delta E = 2\chi \,, \quad \delta C = 2\Delta\chi \,, \quad \delta \widehat{h}_{ij} = 0 \,.
\end{aligned}
$$

Gauge invariant combinations (Bardeen variables):

$$
\Sigma_i \equiv \dot{E}_i - B_i \,, \quad \Phi \equiv \tfrac{1}{3}(C - \Delta E) \,, \quad \Psi \equiv \phi + \dot{B} - \tfrac{1}{2}\ddot{E}
$$

Gravity on Flat Space

Einstein equations:

$$G_{00} = -\Delta\Phi\,,$$
$$G_{0i} = \frac{1}{2}\Delta\Sigma_i - \partial_i\dot{\Phi}\,,$$
$$G_{ij} = -\frac{1}{2}\Box\,\widehat{h}_{ij} - \partial_i\partial_j\Big(\Psi + \frac{1}{2}\Phi\Big) + \delta_{ij}\Big(\Delta\Psi + \frac{1}{2}\Delta\Phi - \ddot{\Phi}\Big) + \partial_{(i}\dot{\Sigma}_{j)}\,.$$

Reduce to:

$$\Box\widehat{h}_{ij} = 0\,.$$

Rewrite Fierz-Pauli action in manifest gauge invariant form:

$$S = \int d^4x\big\{\tfrac{1}{4}\widehat{h}^{ij}\,\Box\,\widehat{h}_{ij} - \tfrac{1}{2}\Sigma^i\Delta\Sigma_i - \tfrac{1}{2}(4\Psi + \Phi)\Delta\Phi + \tfrac{3}{2}\Phi\ddot{\Phi}\big\}$$

Gravity on FRW Backgrounds

Einstein-Hilbert Action with Minimally Coupled Scalar Field:

$$S = \int d^4x \sqrt{-g} \left\{ R - \tfrac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right\},$$

Expand around FRW:

$$\begin{aligned} g_{\mu\nu}(\eta, x) &= a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu}(\eta, x)), \\ \varphi(\eta, x) &= \varphi_0(\eta) + \varphi_1(\eta, x). \end{aligned}$$

Gauge invariant variables:

$$\begin{aligned} \Sigma_i &= \dot{E}_i - B_i, \\ \Phi &= -C + \frac{1}{3}\Delta E - H(B - \dot{E}), \\ \Psi &= \phi + H(B - \dot{E}) + \dot{B} - \ddot{E}, \\ \Theta &= \varphi_1 + \dot{\varphi}_0(B - \dot{E}). \end{aligned}$$

Gravity on FRW Backgrounds

After lots of computation:

$$
\begin{aligned}
S \ = \ \int d^4x \ a^2 \Bigg\{ & \frac{1}{4}\dot{\widehat{h}}^{ij}\dot{\widehat{h}}_{ij} + \frac{1}{4}\widehat{h}^{ij}\Delta\widehat{h}_{ij} - \frac{1}{2}\Sigma_i\Delta\Sigma^i \\
& + 4\Phi\Delta\Psi - 2\Phi\Delta\Phi - 6(\dot{\Phi} + H\Psi)^2 + \frac{1}{2}\dot{\varphi}_0{}^2\Psi^2 \\
& + \frac{1}{2}\dot{\Theta}^2 + \frac{1}{2}\Theta\Delta\Theta - \frac{1}{2}a^2 V''(\varphi_0)\Theta^2 \\
& + \dot{\varphi}_0\Theta(\dot{\Psi} + 3\dot{\Phi}) - 2a^2 V'(\varphi_0)\Theta\Psi \Bigg\} \, .
\end{aligned}
$$

Questions:

- How to discern propagating degrees of freedom?
- How to expand to cubic order (and beyond!) in an efficient way?