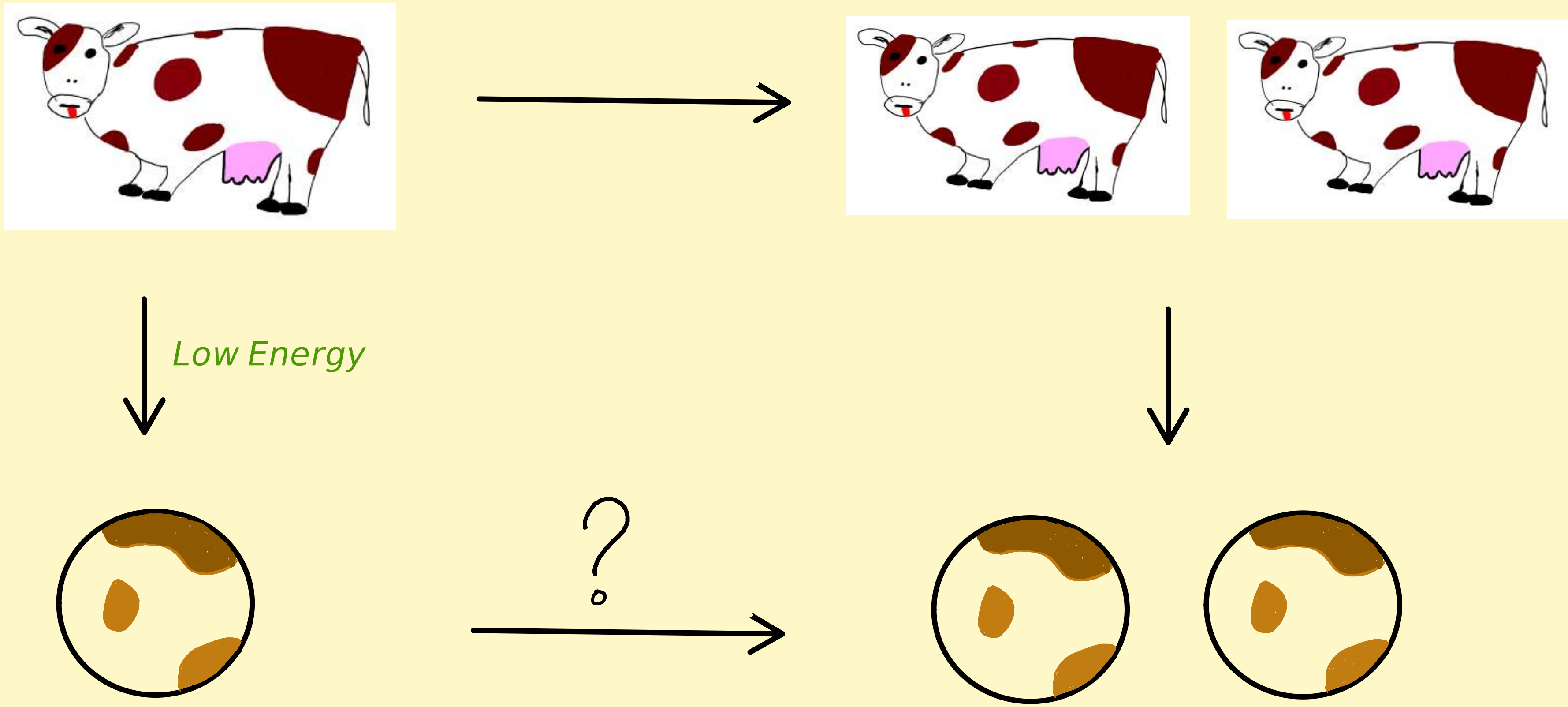
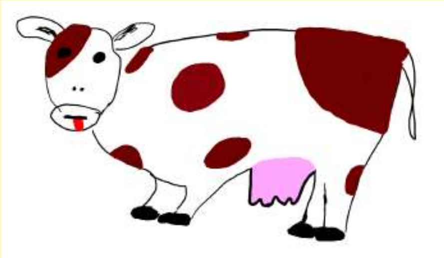
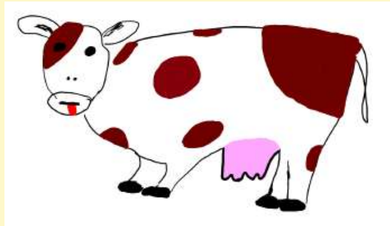
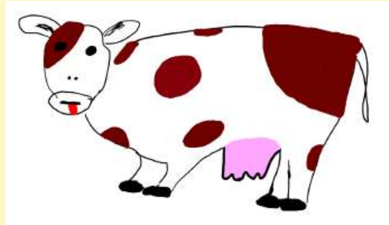
Student seminar - 16.02.2021

Jasper Roosmale Nepveu



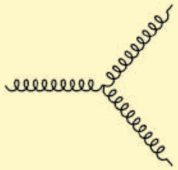
...explained to Daniele's grandmother

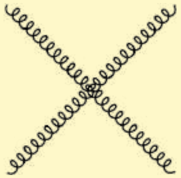


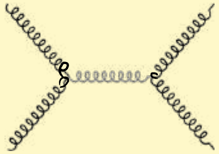
 

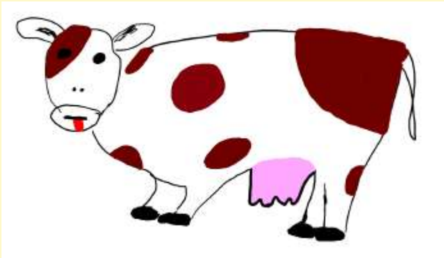
$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\mathcal{L}_{\mathrm{EH}+\phi+B} = \sqrt{-g}\left[-\frac{1}{2}R + \frac{1}{4}\partial^\mu\phi\partial_\mu\phi + \frac{1}{6}\mathrm{e}^{-2\phi}H^{\lambda\mu\nu}H_{\lambda\mu\nu}\right]$$

 $\sim \; gf^{abc}$

 $\sim \; g^2 f^{abx} f^{xcd}$ + Perms

$\sim$  + perms



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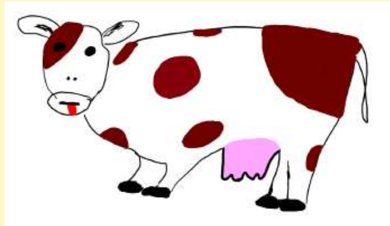 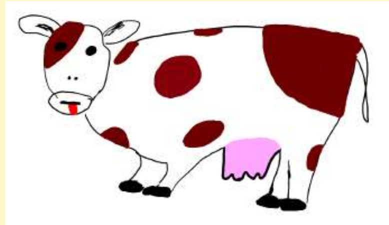

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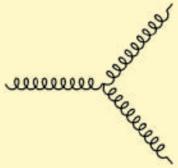
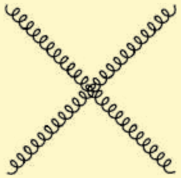
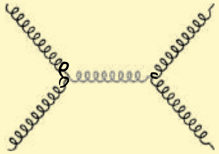
 $\sim \; g f^{abc}$

 $\sim \; g^2 f^{abx} f^{xcd}$ + Perms

$\sim$  + perms

$$\Rightarrow \quad \mathcal{A}^{\text{tree}} = \sum_{i-\substack{\text{trivalent}\\\text{graphs}}} \frac{c_i\, n_i}{D_i}$$

L loops:

$$\mathcal{A}^{\text{loop}} = \sum_{i-\substack{\text{trivalent}\\\text{graphs}}} \int \prod_{l=1}^{L} \frac{d^4 k_l}{(2\pi)^4} \frac{1}{S_i} \frac{c_i\, n_i(k_l)}{D_i(k_l)}$$

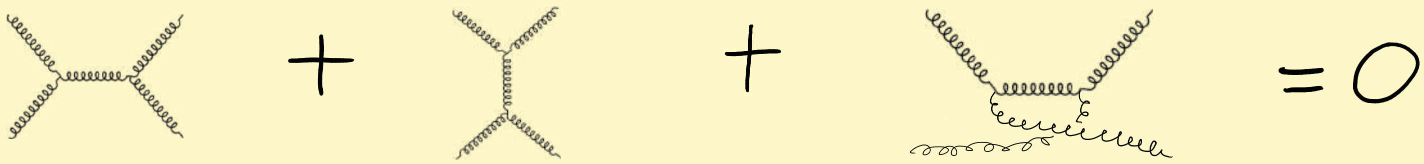
Colour-Kinematics Duality

$$\mathcal{A} = \sum_{i-\substack{\text{trivalent}\\ \text{graphs}}} \frac{c_i\, n_i}{D_i}$$

Colour factors c_i are antisymmetric and obey Jacobi identities:

$$f^{abc} = -f^{acb}$$

$$f^{abx} f^{xcd} + f^{acx} f^{xdb} + f^{adx} f^{xbc} = 0$$



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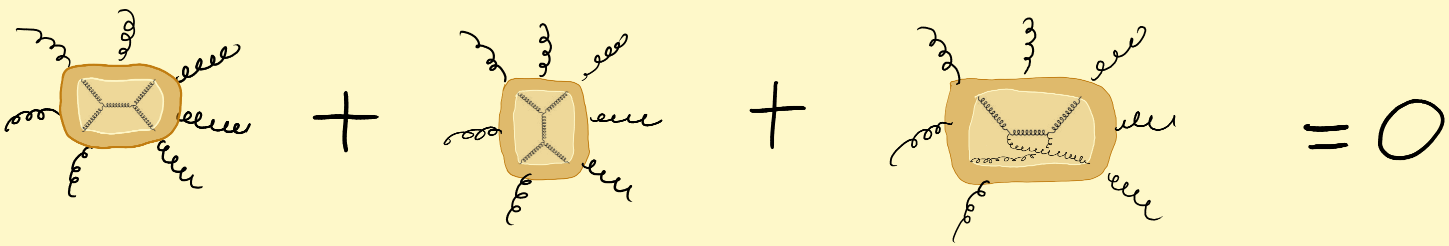

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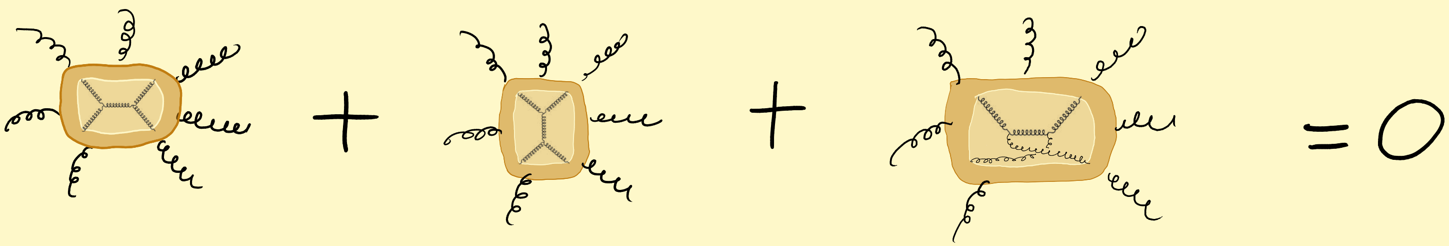

Observation: kinematic numerators factors n_i can be arranged to obey the same relations:

$$\sum_i c_i = 0 \quad \Leftrightarrow \quad \sum_i n_i = 0$$

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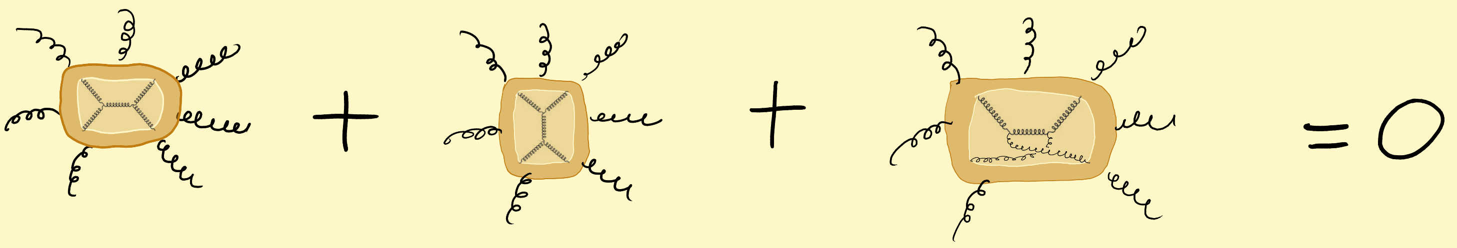

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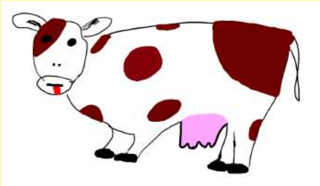
$$\sum_i c_i = 0 \quad \Leftrightarrow \quad \sum_i n_i = 0$$

$\Rightarrow$

- *BCJ relations between colour-ordered amplitudes*
- *Construct gravity amplitudes* $\mathcal{M} = \sum_{i-\substack{\text{trivalent}\\ \text{graphs}}} \frac{n_i\, n_i}{D_i}$



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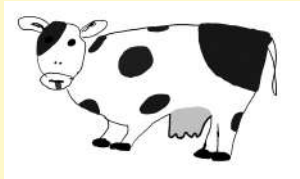 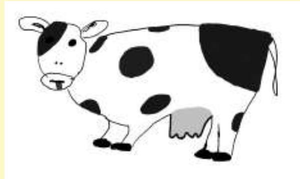

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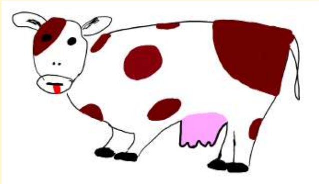
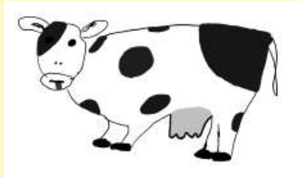
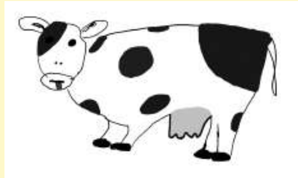
Why is this a gravity amplitude?



$\longrightarrow$

$$\mathcal{A} = \sum_{i-\text{trivalent graphs}} \frac{c_i\, n_i}{D_i}$$

$$\mathcal{M} = \sum_{i-\text{trivalent graphs}} \frac{n_i\, n_i}{D_i}$$

Why is this a gravity amplitude?

Spin 2 : $\partial^+_\mu \partial^+_\nu \to \partial^{++}_{\mu\nu}$

Poles correspond to physical particles because $\dfrac{1}{D_i}$ are unchanged

Gauge invariance :

Gluons: $\varepsilon_\mu(p) \to \varepsilon_\mu(p) + p_\mu$, $\quad n_i \to n_i + \partial_i$

$$\mathcal{A} = \sum_i \frac{c_i\, n_i}{D_i} - \sum_i \frac{c_i\, n_i}{D_i} + \sum_i \frac{c_i\, \varepsilon_i}{D_i}$$

$\to$ vanishes!

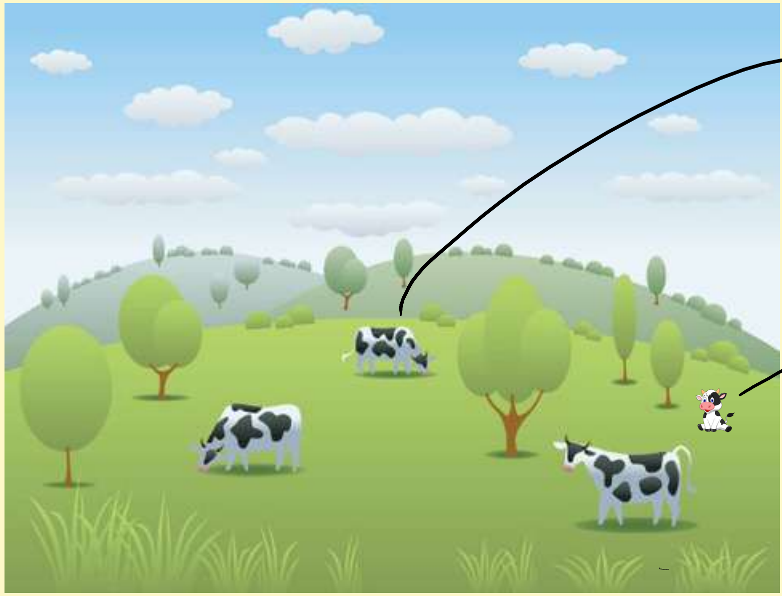
$\Rightarrow$ By CK duality:

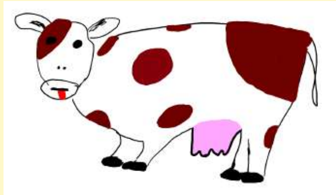
$$\sum_i \frac{n_i\, \delta_i}{D_i} = 0$$

$\to$ Gravitons: $\varepsilon_{\mu\nu}(p) = \varepsilon_\mu(p)\varepsilon_\nu(p) \to \varepsilon_\mu(p)\varepsilon_\nu(p) + \varepsilon_\mu p_\nu + p_\mu \varepsilon_\nu$

$$\mathcal{M} \to \sum_i \frac{n_i\, n_i}{D_i} + \sum_i \frac{n_i\, \delta_i}{D_i} + \sum_i \frac{\delta_i\, n_i}{D_i}$$

How general is the double copy prescription?



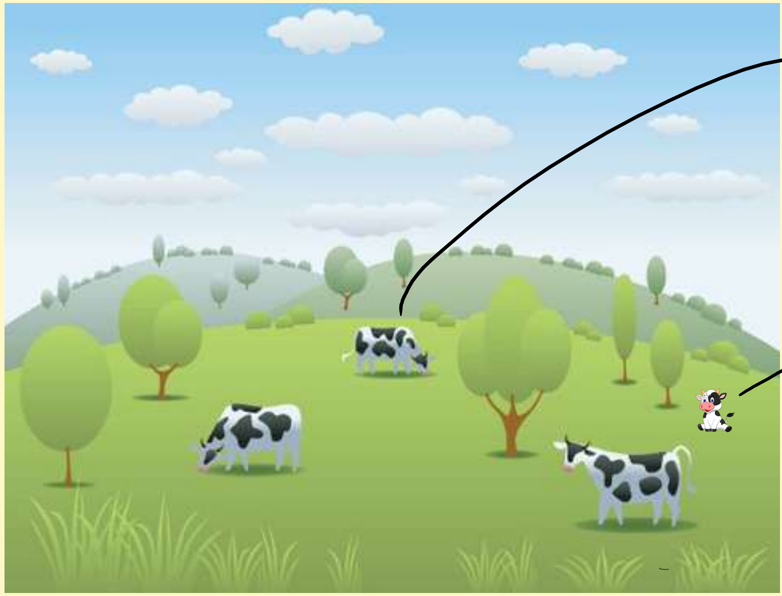


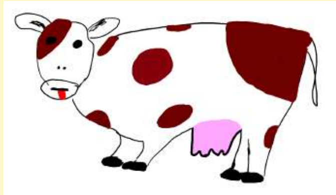
$$\mathcal{L}_{\rm YM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^{a{\rm A}} \partial^\mu \phi^{a{\rm A}} + \frac{g}{3!} f^{abc} F^{\rm ABC} \phi^{a{\rm A}} \phi^{b{\rm B}} \phi^{c{\rm C}}$$

(can be obtained from replacing n_i by c_i in YM amplitudes)

How general is the double copy prescription?





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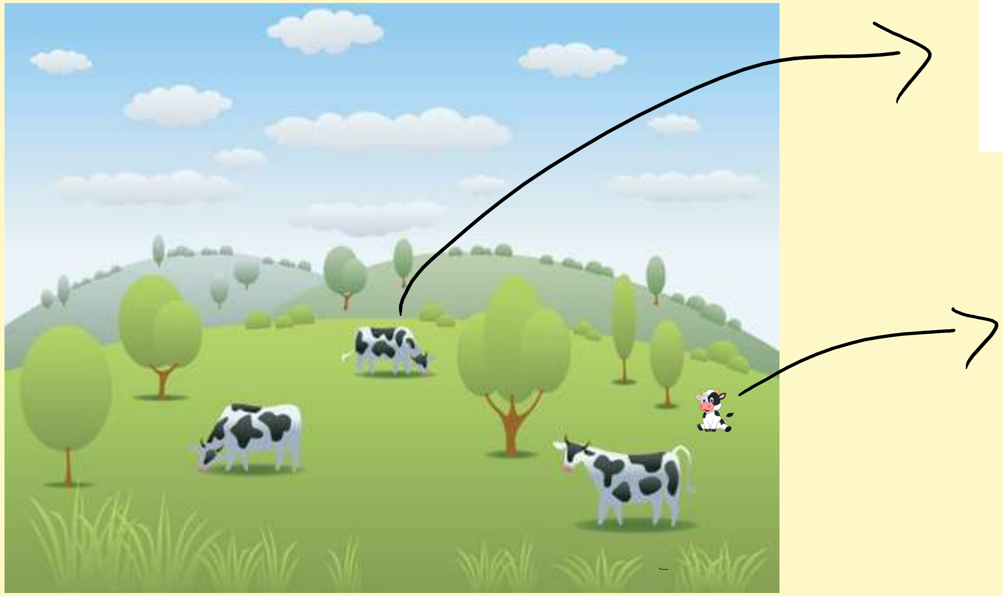
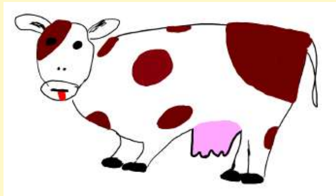
$$\mathcal{L}_{\mathrm{YM}+\phi} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\delta^{a\mathrm{A}}D^\mu\delta^{a\mathrm{A}} - \frac{g^2}{4}f^{abx}f^{xcd}\delta^{a\mathrm{A}}\delta^{b\mathrm{B}}\delta^{c\mathrm{A}}\delta^{d\mathrm{B}}$$ [1511.01740]

$$\mathcal{L}_{\mathrm{QCD}} = \mathcal{L}_{\mathrm{YM}} + \overline{\partial}(\not{\partial} - m)\partial$$ [1507.00332]

$$\mathcal{L}_{s\mathrm{QCD}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varepsilon_i)^\dagger D^\mu\varepsilon_i - m^2\varepsilon_i^\dagger\varepsilon_i$$ [1911.06785] & [2010.13435]

How general is the double copy prescription?





$$\mathcal{L}_{\rm YM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^{a\rm A}\partial^\mu\phi^{a\rm A} + \frac{g}{3!}f^{abc}F^{\rm ABC}\phi^{a\rm A}\phi^{b\rm B}\phi^{c\rm C}$$

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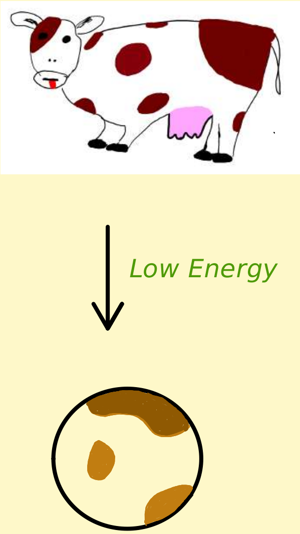
$$\mathcal{L}_{\rm YM+\phi} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\delta^{a\rm A}D^\mu\delta^{a\rm A} - \frac{g^2}{4}f^{abx}f^{xcd}\delta^{a\rm A}\delta^{b\rm B}\delta^{c\rm A}\delta^{d\rm B}$$ [1511.01740]

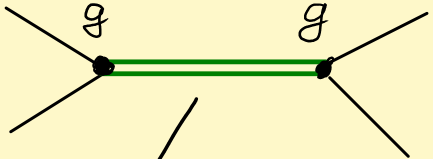
$$\mathcal{L}_{\rm QCD} = \mathcal{L}_{\rm YM} + \overline{\partial}(\not{\partial} - m)\partial$$ [1507.00332]

$$\mathcal{L}_{s\rm QCD} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varepsilon_i)^\dagger D^\mu\varepsilon_i - m^2\varepsilon_i^\dagger\varepsilon_i$$ [1911.06785]

Asks for a systematic approach! ➡ EFT

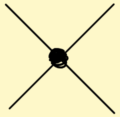
Effective Field Theory

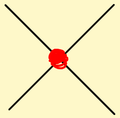


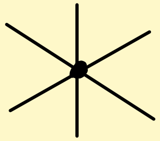


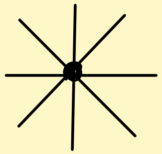
$$\frac{1}{p^2-m^2} \approx \frac{-1}{m^2}\left(1+\frac{p^2}{m^2}+\ldots\right)$$

Low Energy

 $= \frac{g^2}{m^2}$

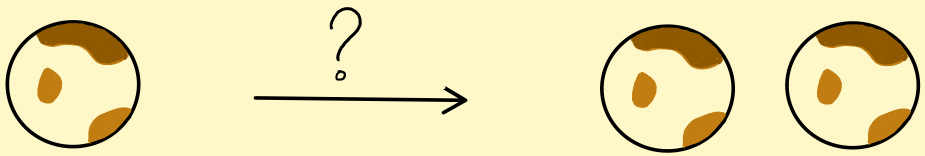
 $= \frac{\mathcal{O}(p^2)}{m^4}$

, 

,  , ...

Parametrise the high energy physics with minimal assumptions
by writing down the most general Lagrangian with free parameters.
Or better: write down the most general amplitude possible!

Examples:



[1208.0876]: from string theory

$$\mathcal{L} = \mathcal{L}_{\rm YM} + \frac{1}{\Lambda^2} F^3 + \frac{1}{\Lambda^4} F^4 + \dots$$

$$F^3 \quad \mathrm{Tr}(T^a T^b T^c) F^a{}_\mu{}^\nu F^b{}_\nu{}^\rho F^c{}_\rho{}^\mu - f^{abc} F^a{}_\mu{}^\nu F^b{}_\nu{}^\rho F^c{}_\rho{}^\mu \quad \checkmark \quad \Rightarrow R + e^{i\phi} R^2 + R^3$$

$$F^4 = \mathrm{Tr}(T^a T^b T^c T^c) F^a F^b F^c F^d \quad \times$$

$$\frac{1}{\Lambda^6} D^2 F^4 + F^5 \quad \checkmark$$

[2009.00008]: build amplitudes at higher orders from lower orders (NLSM)

at 4pt:

$$\left.\begin{aligned} c_s + c_t + c_u &= 0 \\ n_s + n_t + n_u &= 0 \end{aligned}\right\} \qquad \frac{1}{\Lambda^2}(s^2 + t^2 + u^2)(n_s + n_t + n_u) = 0$$

$$\left.\begin{aligned} J_s &= j_t j'_t - j_u j'_u \\ J_t &= j_u j'_u - j_s j'_s \\ J_u &= j_s j'_s - j_t j'_t \end{aligned}\right\} J_s + J_t + J_u = 0$$

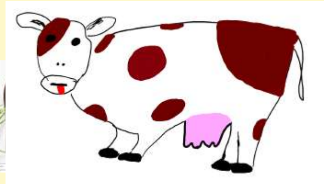
[1910.12850] : F^4 operators allow for a double copy

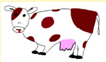
But are we not looking for too much?

- Is the "double copy prescription" closed under renormalisation?

- Can we relax the constraint of Colour-Kinematics duality
  if we assume that the high energy theory satisfies this?

Effective field theory of the bi-adjoint scalar



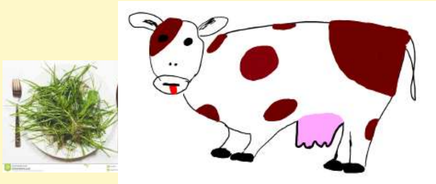


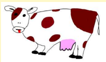
$$\mathcal{L}^{(\mathrm{UV})} = \frac{1}{2}\varepsilon_{\,\mu}\varphi^{a\mathrm{A}}\varepsilon^{\,\mu}\varphi^{a\mathrm{A}} - \frac{1}{2}m^2\varphi^{a\mathrm{A}}\varphi^{a\mathrm{A}} + \frac{g}{3!}f^{abc}F^{\mathrm{ABC}}\delta^{\,a\mathrm{A}}\delta^{\,b\mathrm{B}}\delta^{\,c\mathrm{C}}$$

$$+\frac{1}{2}\partial_\mu\phi^{a\mathrm{A}}\partial^\mu\phi^{a\mathrm{A}} - \frac{1}{2}M^2\phi^{a\mathrm{A}}\phi^{a\mathrm{A}}$$

$$+\frac{g}{2}f^{abc}F^{\mathrm{ABC}}\partial^{a\mathrm{A}}\phi^{b\mathrm{B}}\phi^{c\mathrm{C}}$$

Effective field theory of the bi-adjoint scalar





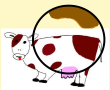
$$\mathcal{L}^{(\mathrm{UV})} = \frac{1}{2}\varepsilon_{\mu}\varphi^{a\mathrm{A}}\varepsilon^{\mu}\varphi^{a\mathrm{A}} - \frac{1}{2}m^2\varphi^{a\mathrm{A}}\varphi^{a\mathrm{A}} + \frac{g}{3!}f^{abc}F^{\mathrm{ABC}}\delta^{\,a\mathrm{A}}\delta^{\,b\mathrm{B}}\delta^{\,c\mathrm{C}}$$

$$+\frac{1}{2}\partial_{\mu}\phi^{a\mathrm{A}}\partial^{\mu}\phi^{a\mathrm{A}} - \frac{1}{2}M^2\phi^{a\mathrm{A}}\phi^{a\mathrm{A}}$$

$$+\frac{g}{2}f^{abc}F^{\mathrm{ABC}}\partial^{a\mathrm{A}}\phi^{b\mathrm{B}}\phi^{c\mathrm{C}}$$

Low Energy

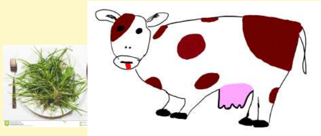
$$\mathcal{L}^{(\mathrm{EFT})} = \frac{1}{2}\varepsilon_{\mu}\varphi^{a\mathrm{A}}\varepsilon^{\mu}\varphi^{a\mathrm{A}} - \frac{1}{2}m^2\varphi^{a\mathrm{A}}\varphi^{a\mathrm{A}}$$

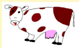


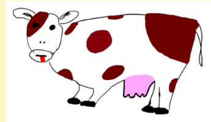
$$+\frac{g}{3!}\left[f^{abc}F^{\mathrm{ABC}} + \frac{1}{4(4\varepsilon)^2}\frac{g^2}{M^2}(f^3)^{abc}(F^3)^{\mathrm{ABC}}\right]\varphi^{a\mathrm{A}}\varphi^{b\mathrm{B}}\varphi^{c\mathrm{C}}$$

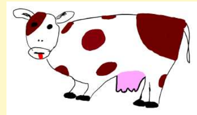
$$+\frac{g}{2}\left[\frac{1}{288(4\delta)^2}\frac{g^2}{M^4}(f^3)^{abc}(F^3)^{\mathrm{ABC}}\right]\varphi^{a\mathrm{A}}\varphi^{b\mathrm{B}}\partial^2\varphi^{c\mathrm{C}}$$

$$+\frac{1}{4!}\left[\frac{1}{6(4\varepsilon)^2}\frac{g^4}{M^4}(f^4)^{abcd}(F^4)^{\mathrm{ABCD}}\right]\varphi^{a\mathrm{A}}\varphi^{b\mathrm{B}}\varphi^{c\mathrm{C}}\varphi^{d\mathrm{D}}$$



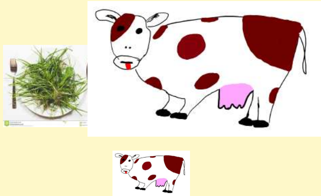


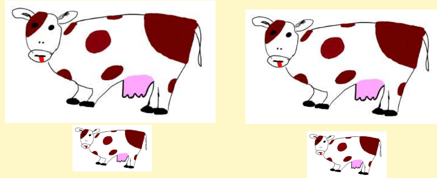




$$
\begin{aligned}
\mathcal{L}^{(\mathrm{UV})} &= \frac{1}{2}\varepsilon_{\mu}\varphi^{a\mathrm{A}}\varepsilon^{\mu}\varphi^{a\mathrm{A}} - \frac{1}{2}m^{2}\varphi^{a\mathrm{A}}\varphi^{a\mathrm{A}} \\
&\quad + \frac{1}{2}\partial_{\mu}\phi^{a\mathrm{A}}\partial^{\mu}\phi^{a\mathrm{A}} - \frac{1}{2}M^{2}\phi^{a\mathrm{A}}\phi^{a\mathrm{A}} \\
&\quad + \frac{g}{3!}f^{abc}F^{\mathrm{ABC}}\delta^{a\mathrm{A}}\delta^{b\mathrm{B}}\delta^{c\mathrm{C}} + \frac{g}{2}f^{abc}F^{\mathrm{ABC}}\partial^{a\mathrm{A}}\phi^{b\mathrm{B}}\phi^{c\mathrm{C}}
\end{aligned}
$$

$$
\mathcal{L}^{(\mathrm{UV})}\Big|_{f-F}
$$



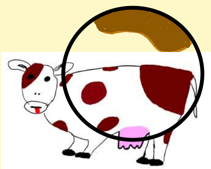


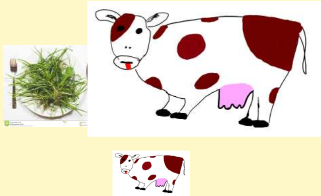
$$\mathcal{L}^{(\mathrm{UV})} = \frac{1}{2}\varepsilon_{\mu}\varphi^{a\mathrm{A}}\varepsilon^{\mu}\varphi^{a\mathrm{A}} - \frac{1}{2}m^2\varphi^{a\mathrm{A}}\varphi^{a\mathrm{A}} + \frac{1}{2}\partial_{\mu}\phi^{a\mathrm{A}}\partial^{\mu}\phi^{a\mathrm{A}} - \frac{1}{2}M^2\phi^{a\mathrm{A}}\phi^{a\mathrm{A}} + \frac{g}{3!}f^{abc}F^{\mathrm{ABC}}\delta^{a\mathrm{A}}\delta^{b\mathrm{B}}\delta^{c\mathrm{C}} + \frac{g}{2}f^{abc}F^{\mathrm{ABC}}\partial^{a\mathrm{A}}\phi^{b\mathrm{B}}\phi^{c\mathrm{C}}$$

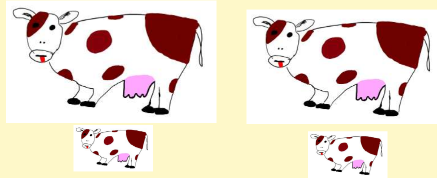
$$\mathcal{L}^{(\mathrm{UV})}\Big|_{f-\,F}$$

Low Energy

$$\mathcal{L}^{(\mathrm{EFT})} = \frac{1}{2}\varepsilon_{\mu}\varphi^{a\mathrm{A}}\varepsilon^{\mu}\varphi^{a\mathrm{A}} - \frac{1}{2}m^2\varphi^{a\mathrm{A}}\varphi^{a\mathrm{A}} + \frac{g}{3!}\left[f^{abc}F^{\mathrm{ABC}} + \frac{1}{4(4\varepsilon)^2}\frac{g^2}{M^2}(f^3)^{abc}(F^3)^{\mathrm{ABC}}\right]\varphi^{a\mathrm{A}}\varphi^{b\mathrm{B}}\varphi^{c\mathrm{C}} + \frac{g}{2}\left[\frac{1}{288(4\delta)^2}\frac{g^2}{M^4}(f^3)^{abc}(F^3)^{\mathrm{ABC}}\right]\varphi^{a\mathrm{A}}\varphi^{b\mathrm{B}}\partial^2\varphi^{c\mathrm{C}} + \frac{1}{4!}\left[\frac{1}{6(4\varepsilon)^2}\frac{g^4}{M^4}(f^4)^{abcd}(F^4)^{\mathrm{ABCD}}\right]\varphi^{a\mathrm{A}}\varphi^{b\mathrm{B}}\varphi^{c\mathrm{C}}\varphi^{d\mathrm{D}}$$







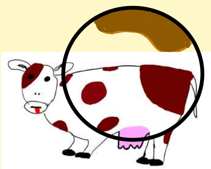
$$\mathcal{L}^{(\mathrm{UV})} = \frac{1}{2}\varepsilon_{\mu}\varphi^{a\mathrm{A}}\varepsilon^{\mu}\varphi^{a\mathrm{A}} - \frac{1}{2}m^2\varphi^{a\mathrm{A}}\varphi^{a\mathrm{A}}$$
$$+\frac{1}{2}\partial_{\mu}\phi^{a\mathrm{A}}\partial^{\mu}\phi^{a\mathrm{A}} - \frac{1}{2}M^2\phi^{a\mathrm{A}}\phi^{a\mathrm{A}}$$
$$+\frac{g}{3!}f^{abc}F^{\mathrm{ABC}}\delta^{a\mathrm{A}}\delta^{b\mathrm{B}}\delta^{c\mathrm{C}} + \frac{g}{2}f^{abc}F^{\mathrm{ABC}}\partial^{a\mathrm{A}}\phi^{b\mathrm{B}}\phi^{c\mathrm{C}}$$

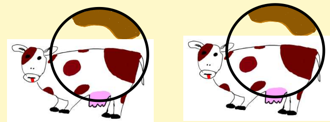
$$\mathcal{L}^{(\mathrm{UV})}\Big|_{f-F}$$

Low Energy

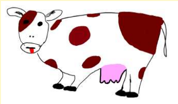
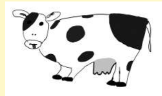
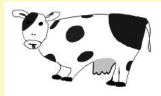
$$\mathcal{L}^{(\mathrm{EFT})} = \frac{1}{2}\varepsilon_{\mu}\varphi^{a\mathrm{A}}\varepsilon^{\mu}\varphi^{a\mathrm{A}} - \frac{1}{2}m^2\varphi^{a\mathrm{A}}\varphi^{a\mathrm{A}}$$
$$+\frac{g}{3!}\left[f^{abc}F^{\mathrm{ABC}} + \frac{1}{4(4\varepsilon)^2}\frac{g^2}{M^2}(f^3)^{abc}(F^3)^{\mathrm{ABC}}\right]\varphi^{a\mathrm{A}}\varphi^{b\mathrm{B}}\varphi^{c\mathrm{C}}$$
$$+\frac{g}{2}\left[\frac{1}{288(4\delta)^2}\frac{g^2}{M^4}(f^3)^{abc}(F^3)^{\mathrm{ABC}}\right]\varphi^{a\mathrm{A}}\varphi^{b\mathrm{B}}\partial^2\varphi^{c\mathrm{C}}$$
$$+\frac{1}{4!}\left[\frac{1}{6(4\varepsilon)^2}\frac{g^4}{M^4}(f^4)^{abcd}(F^4)^{\mathrm{ABCD}}\right]\varphi^{a\mathrm{A}}\varphi^{b\mathrm{B}}\varphi^{c\mathrm{C}}\varphi^{d\mathrm{D}}$$

$$\mathcal{L}^{(\mathrm{EFT})}\Big|_{f-F}$$





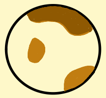
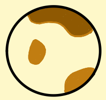
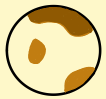
Conclusion

-  $\mathcal{A} = \int_{i-\text{trivalent graphs}} \frac{c_i\, n_i}{D_i} \longrightarrow \mathcal{M} = \int_{i-\text{trivalent graphs}} \frac{n_i\, n_i}{D_i}$  

- Effective field theory allows for a general exploration of theories

- But the "double copy prescription" is not carried over to low energies

⇒ To do: work out a real example, maybe scalar or fermionic QCD:

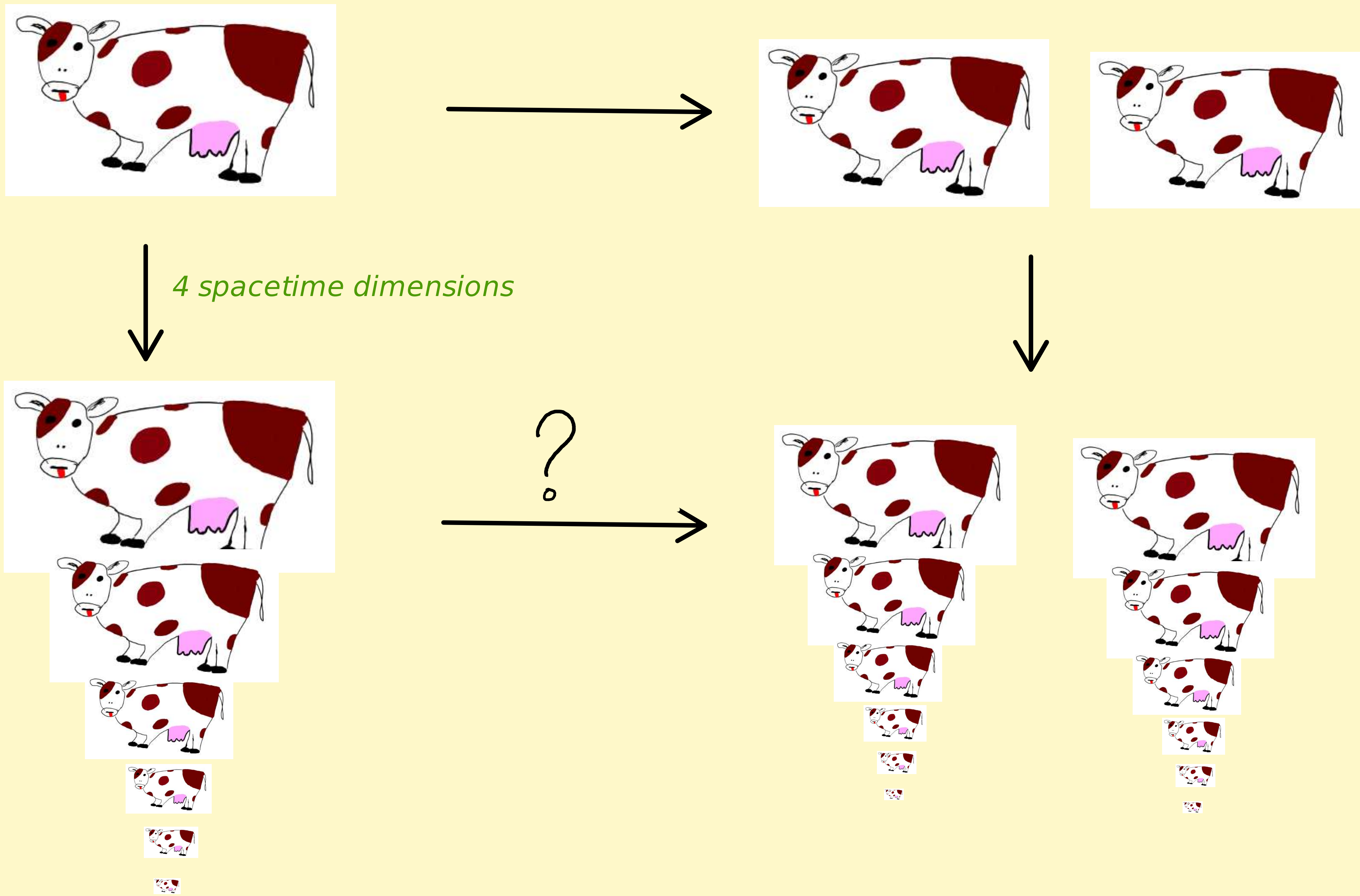
Find a prescription for  $\xrightarrow{?}$  

- Interesting whether this exists and what it looks likes
- Possible phenomenological advantages in calculations (?)

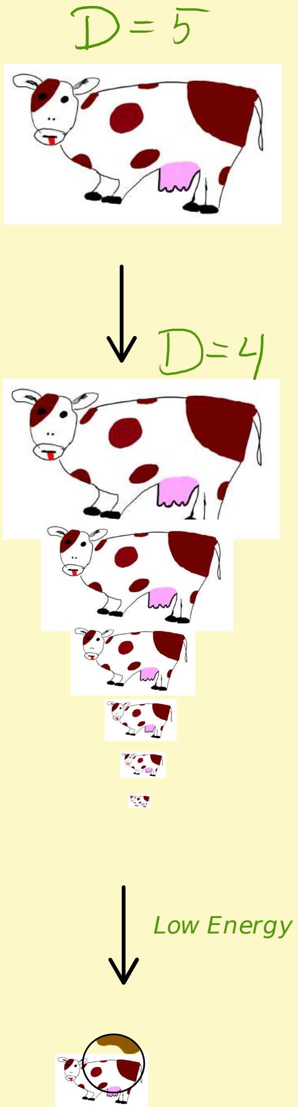
Thank you!

Extra slides

Kaluza-Klein dimensional reduction



Kaluza-Klein dimensional reduction of the bi-adjoint scalar



$$\mathcal{L}^{(5)} = \frac{1}{2}\partial_\mu \phi^{aA}\partial^\mu \phi^{aA} - \frac{1}{2}\partial_5 \phi^{aA}\partial_5 \phi^{aA} + \frac{g}{3!} f^{abc} F^{ABC} \phi^{aA}\phi^{bB}\phi^{cC}$$

$$S = \int d^4x \underbrace{\int_0^{2\pi R} dy\, \mathcal{L}^{(5)}}_{\mathcal{L}^{(4)}}$$

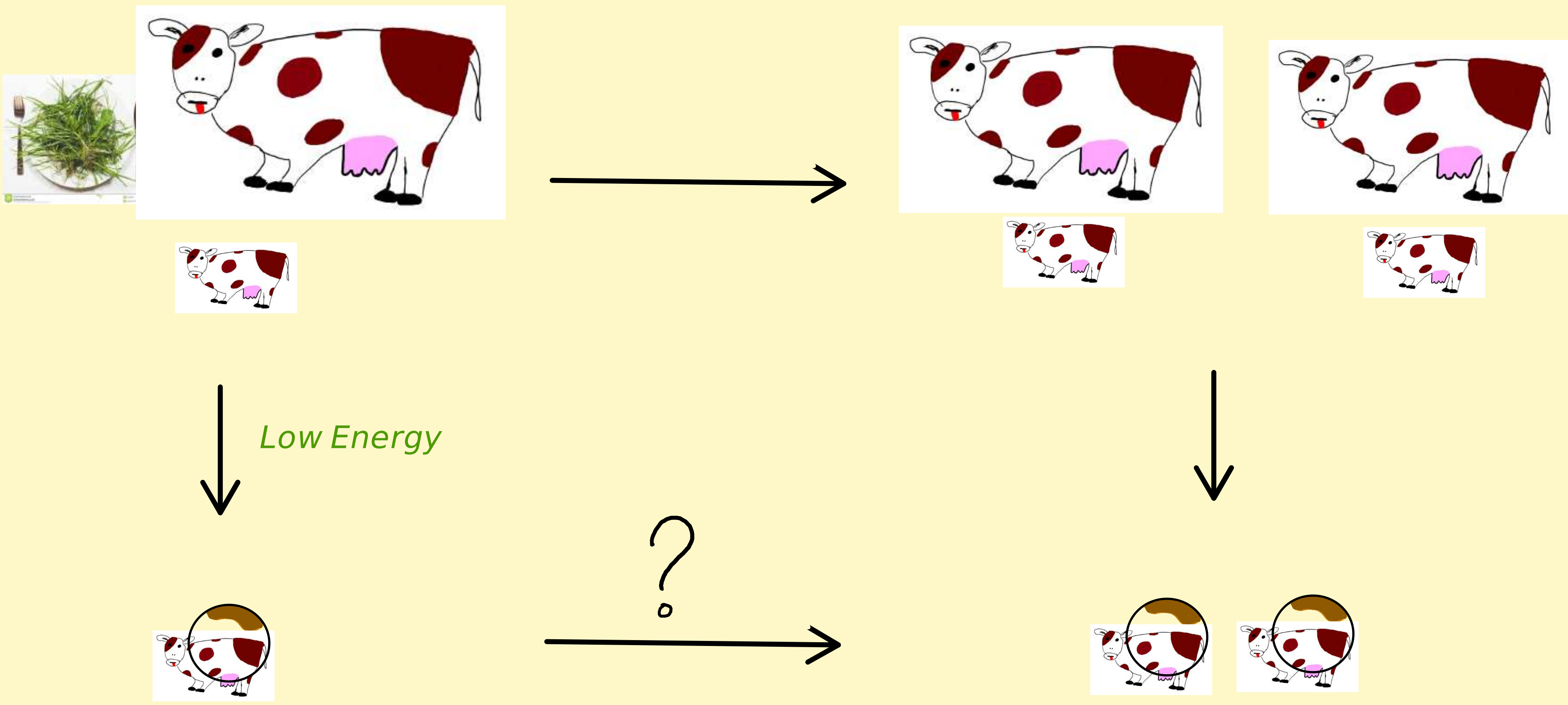
$$\phi = \phi(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{\infty} \varphi^{(n)}(x)\, e^{iny/R}$$

$$\left(\varphi^{(n)} = \varphi^{(-n)\dagger}\right)$$

$$\mathcal{L}^{(4)} = \frac{1}{2}\partial_\mu \varphi_0^{aA}\partial^\mu \varphi_0^{aA} + \sum_{n=1}^{\infty} \left[\partial_\mu \varphi_n^{aA}\partial^\mu \varphi_{-n}^{aA} - M_n^2 \varphi_n^{aA}\varphi_{-n}^{aA}\right]$$

$$+ \frac{g}{3!} f^{abc} F^{ABC} \sum_{n,m,l=-\infty}^{\infty} \varphi_n^{aA}\varphi_m^{bB}\varphi_l^{cC}\, \delta^0_{n+m+l}$$

$$\hookrightarrow M_n = \frac{n}{2\pi R}$$



Need a massive theory that can be "double copied"

$$
\begin{aligned}
(\varepsilon^{h})^{ij}_{\mu\nu} &= \varepsilon^{((i}_{\mu}\varepsilon^{j))}_{\nu} && \text{(graviton)}\,, \\
(\varepsilon^{B})^{ij}_{\mu\nu} &= \varepsilon^{[i}_{\mu}\varepsilon^{j]}_{\nu} && (B\text{-field})\,, \\
(\varepsilon^{\phi})_{\mu\nu} &= \frac{\varepsilon^{i}_{\mu}\varepsilon^{j}_{\nu}\delta_{ij}}{D-2} && \text{(dilaton)}\,.
\end{aligned}
$$