Wittgenstein's *ab*-Notation: An Iconic Proof Procedure

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Abstract

This paper systematically outlines Wittgenstein's *ab*-notation. The purpose of this notation is to provide a proof procedure in which ordinary logical formulas are converted into ideal symbols that identify the logical properties of the initial formulas. The general ideas underlying this procedure are in opposition to a traditional conception of axiomatic proof and are related to Peirce's iconic logic. Based on Wittgenstein's scanty remarks concerning his *ab*-notation, which almost all apply to propositional logic, this paper explains how to extend his method to a subset of first-order formulas, namely, formulas that do not contain dyadic sentential connectives within the scope of any quantifier.

Keywords: Wittgenstein; iconic logic; symbolic logic; first-order logic; logical proof

1 Introduction

Wittgenstein's contribution to logic in the *Tractatus* is often associated with truth tables. However, it is rarely noted that Wittgenstein never used his 'TF-schemata' as a decision procedure for propositional formulas. Instead, he designed his so-called *ab*-notation for this purpose, between 1912 and 1914. At that time, he was also confident in applying this notation to the entire realm of first-order logic (FOL).[1] Unlike truth tables, the *ab*-notation contains quantifiers.[2] However, Wittgenstein never seriously attempted to spell out his *ab*-notation for FOL or for parts thereof. In the *Tractatus*,

[1] Cf. CL, letter 30 (p. 53) and letter 32 (p. 57).
[2] Cf. NL (p. 96[1]).

he described a method similar to the *ab*-notation to illustrate his idea of logical proof that allows one to identify logical properties such as logical truth 'from the symbol alone' (TLP 6.113). However, the method described in the *Tractatus* applies solely to propositional logic.[3] In contrast to his work during the period from 1912 to 1914, in the *Tractatus*, he defined generality in terms of infinite sets of propositions.[4] This might have abated his interest in a notation that does not eliminate quantifiers. Wittgenstein later disavowed his reductive analysis of quantifiers as 'the biggest mistake' of the *Tractatus* (*von Wright* (*1982*, p. 151)). Consequently, he returned to a non-reductive analysis of quantified formulas.[5] Yet, he never took up his initial *ab*-notation again, nor did he ever illustrate in detail how his idea of logical proof applies to FOL.

It is often recognised that Wittgenstein's conception of proof is incompatible with the Church-Turing theorem. *Landini* (*2007*, pp. 112–118) accurately observes that Wittgenstein's idea of a logical proof implies a decision procedure in terms of an algorithmic translation of FOL formulas into symbols of an ideal notation 'in which all and only logical equivalents have exactly one and the same expression' (p. 112). The Church-Turing theorem implies that this is impossible for the entire realm of FOL. Consequently, most scholars have rejected Wittgenstein's conception of logic in general and his *ab*-notation in particular because Wittgenstein's programmatic claim that his idea of proof applies to FOL appears to be refuted by modern logic.[6]

However, although Wittgenstein did not present any details regarding how to realise his conception of proof in the realm of FOL and its full realisation in that realm seems utopian, his basic idea of a procedure for translating ordinary FOL formulas into symbols of a notation that provides criteria for identifying the logical properties of the initial formulas is rather clear-cut. Such a procedure is an alternative to an axiomatic proof procedure that derives theorems from axioms within a correct and complete calculus. Contrary to a traditional conception of logic, Wittgenstein's approach does not separate a theory of deduction from the semantics that serve as a standard for the correctness and completeness of the calculus. Instead, the calculus is replaced by a translation algorithm that is intended to interpret initial

[3]Cf. TLP 6.1203.

[4]Cf. TLP 5.5-5.503, 5.52, 6.

[5]Cf. VW (pp. 162-170, cf. in particular p. 165: 'the concept "all" is a primitive concept').

[6]Cf. most recently *Floyd 2005*, p. 95, *Landini 2007*, p. 118, and *Potter 2009*, pp. 181–182.

formulas by means of resulting *ideal symbols* that identify the conditions for truth and falsehood and, consequently, identify the logical properties of instances of the initial formula.

As noted by *Potter* (*2009*, p. 182), Wittgenstein's conception of logic leaves no room for understanding a formula independently of understanding its logical properties. For Wittgenstein, understanding logical formulas is not like understanding an ordinary question without knowing its answer. Instead, it is rather like understanding the sense of a sentence by understanding its inferential relations to other sentences. One does not understand logical formulas if one cannot make judgements regarding their logical equivalence. Thus, paraphrases of or reading algorithms for ordinary formulas as well as semantics that interpret formulas in accordance with their logical hierarchy with respect to some given interpretation (such as model theory) become obsolete from this perspective. The same applies to a proof procedure that identifies no more than theoremhood without elucidating logical properties according to the syntactic properties of symbols. Instead, Wittgenstein endorses a translation procedure based on nothing but equivalence transformation, leading to unambiguous representatives of classes of equivalent formulas that serve to answer all relevant logical questions. Thus, even if one does not believe in the realisability of Wittgenstein's approach to logic within the entire realm of FOL, it is interesting to study it in more detail when comparing alternative approaches to logic.

2 Wittgenstein and Peirce

Wittgenstein's approach might best be understood as a variant of an iconic logic that is placed in opposition to traditional symbolic logic. This distinction, and the most prominent and well-elaborated version of such an iconic logic, is related to Peirce's existential graphs, which have been studied in considerable detail recently.[7] Peirce distinguishes two purposes of logic: to investigate logical theories and to aid in the drawing of inferences.[8] A logical calculus serves the latter purpose, whereas a logical system serves the former. Such a system should explain what is expressible by means of logic. To this end, it must not allow for 'any superfluity of symbols' (*Peirce*, (*1931-1958*, 4.373)):

> It should be recognised as a defect of a system intended for logical study that it has two ways of expressing the same fact, or any

[7]Cf., in particular, *Shin 2002* and *Dau 2006*.
[8]Cf. *Peirce* (*1931-1958*, 4.373).

> superfluity of symbols, although it would not be a serious defect for a calculus to have two ways of expressing a fact.

Wittgenstein did not explicitly distinguish these two purposes of logic as Peirce did. However, similar to Peirce's distinction between the calculi of symbolic logic and his existential graphs, Wittgenstein drew a distinction between the axiomatic proof method and his own proof method.[9] On the one hand, he emphasised that the two methods are equivalent (i.e., do not differ in their results).[10] On the other hand, he regarded the traditional method of symbolisation that allows for 'a plurality' of equivalent symbols as defective as soon as one considers the analysis of propositions (NL (p. 102[3]); see also NL (p. 93[1]) and TLP 5.43):

> If $p = not - not - p$ etc.; this shows that the traditional method of symbolism is wrong, since it allows a plurality of symbols with the same sense; and thence it follows that, in analysing such propositions, we must not be guided by Russell's method of symbolising.

Iconic logic might be distinguished from symbolic logic by the search for a translation procedure to a proper, unambiguous symbolism that does not permit any 'plurality' or 'superfluity' of symbols and, thus, allows one to decide logical problems by means of the properties of that symbolism. In this respect, Wittgenstein's account of logic is iconic, and his *ab*-notation is designed to satisfy the purposes of an iconic logic.

Wittgenstein regarded the need for a theory of deduction and for semantics that extend beyond pure logic as a result of a deficient symbolism that makes it impossible to identify 'the sense' of propositions (i.e., the conditions for or possibilities of their truth and falsehood) by means of the symbolic features of an unambiguous symbolism that does not allow for different yet logically equivalent expressions. According to Wittgenstein, it is not reality (facts) but the *logical possibilities* of truth and falsehood (logical pictures of facts) that are represented by propositions that instantiate logical formulas. Logical formulas concern the logical form of such a representation, and this form is represented by ideal symbols of a proper notation. Logical properties (such as being tautologous or contradictory) and logical relations (such as logical implication or equivalence) follow from the identification of the logical possibilities of the truth and falsehood of propositions. The purpose of Wittgenstein's *ab*-notation is to identify logical properties and relations by means of a proper symbolism. Wittgenstein's *ab*-notation is based on the principle of bipolarity, which calls for some sort of symbolic symmetry to

[9]Cf. MN (p. 109), TLP 6.125.

[10]Cf. TLP 6.125f. and WVC (p. 80).

represent the difference between the possibilities of truth *and* of falsehood. In this respect, Wittgenstein's *ab*-notation differs from Peirce's existential graphs, which rather seem to be guided by the idea of representing real facts.[11] More generally, Wittgenstein's account of logic differs from Peirce's approach in that it calls for bipolarity as a fundamental property of a proper logical notation, whereas Peirce claims that 'symmetry always involves superfluity' and that symmetries 'are great evils' for 'the purposes of analysis' (*Peirce* (*1931-1958*, 4.375)).

Given that the purpose of iconic logic is to provide a tool for analysing instances of logical formulas (propositions) 'by finding a form of representation in which all and only logical equivalents have exactly one and the same expression' (*Landini*, (2007, p. 112)), designing an 'iconic' proof (or even decision) procedure in terms of an algorithmic translation of logical formulas into their representatives in a proper notation (including a final reading algorithm for those representatives) is the ultimate goal of this logic. Based on this understanding, it is remarkable that neither Peirce's graphs nor Wittgenstein's remarks on his *ab*-notation achieve this goal. This is even true in the case of propositional logic. To my knowledge, specifying such a translation procedure for a relevant subset of FOL is still a desideratum of logical studies.[12]

Wittgenstein motivated his conception of proof philosophically. He looked for a conception of logic that reduces the study of logical properties to pure formal properties that can be identified from properties of a proper symbolism. However, he was simply not interested in doing the necessary logical work to explicitly express his programmatic claims. This work must still be done if one wishes to seriously discuss his understanding of logic. The aim of this paper is to define Wittgenstein's *ab*-notation for a relevant fragment of FOL in terms of a translation procedure that translates all logically equivalent ordinary FOL formulas into one and only one representative within the *ab*-notation. As a starting point, I will achieve this for propositional logic (section 3). I will then extend the procedure to what I call 'elementary FOL',

[11] Cf. *Shin 2002*, p. 52.

[12] Shin *2002*, p. 93 notes that the 'work on EG [existential graphs] has concentrated on translating from graphs of EG to symbolic languages'. This is also true of her own work. However, she also 'reverses the traditional relationship' (ibid) by specifying an algorithm for translating propositional formulas into α-graphs. Unfortunately, she does not discuss an analogous translation procedure for first-order formulas (with identity) and β-graphs. Moreover, although she notes that Peirce's existential graphs assign the same graphs to *some* equivalent formulas (p. 95), she does not discuss procedures for transforming existential graphs such that *all* equivalent formulas of some relevant part of FOL are related to one and the same graph.

i.e., FOL formulas (without identity) that do not contain dyadic sentential connectives within the scope of quantifiers (section 4). Within this fragment of FOL, not only is it decidable whether a given formula is logically true (or false), but one can also nicely demonstrate (i) how to achieve representatives that allow one to read off the conditions for truth and falsehood of instances of the initial formulas for each class of equivalent formulas and (ii) how to analyse the logical relations between different equivalence classes by means of relations between the syntactic properties of their representatives.

3 Propositional Logic

TLP 6.1203 describes a method for 'recogniz[ing] an expression as a tautology' that differs from the *ab*-notation in only two respects: (i) T and F are used instead of a and b as the poles of propositional variables, and (ii) it is explicitly restricted to propositional logic. I begin the discussion of Wittgenstein's *ab*-notation by describing this method before discussing how it can be generalised to elementary FOL. As elementary FOL is the more general case, I will specify the necessary rules explicitly in section 4, whereas in this section, I will focus on basic features of the *ab*-notation as they arise from Wittgenstein's remarks.

In TLP 6.1203, Wittgenstein basically describes how to translate a given formula into a diagram and how to decide, based on properties of the diagram, whether the initial formula is a tautology. As we will see in this section, Wittgenstein's diagrams are rather cumbersome compared with truth tables or disjunctive normal forms. Section 4, however, will show that the complexity of Wittgenstein's diagrams is due to the intention to extend the *ab*-notation to FOL.

Wittgenstein draws a significant distinction between the properties of a given *ab*-diagram in general and its 'symbolising' properties.[13] Only symbolising properties are significant as the identity criteria for logical properties. *ab*-diagrams, as well as formulas and *signs*, may, in general, differ. However, all equivalent formulas are represented by one and the same *symbol*, which must be read from the *ab*-diagram. To identify 'the symbol' that is represented by an *ab*-diagram, I distinguish between *ab*-diagrams and *ab*-symbols. I will describe a procedure for generating *ab*-symbols from formulas via *ab*-diagrams. Let us begin with the construction of diagrams from propositional formulas, as described by Wittgenstein in TLP 6.1203.[14]

[13]Cf. NL (p. 99[2]) and MN (p. 115).

[14]In addition to TLP 6.1203, one can find three *ab*-diagrams of propositional logic in

3.1 *ab*-Diagrams

To convert a formula into its *ab*-diagram, the formula must be parsed from inside to outside in accordance with its logical hierarchy. First, each occurrence of a propositional variable is provided with an *a*-pole to the left and a *b*-pole to the right. Therefore, instead of p, one must write apb (CL (letters 28 and 32)), or, similarly, $a - p - b$ (e.g., NL (pp. 94[6] and 106[3]), MN (p. 114f.), and CL (letter 28)). The position of the poles is not significant.[15] However, it is important that *two* poles are assigned. This makes it explicit that propositions differ in form from names. Bipolarity is the symbolic criterion for the *logical possibility* that a proposition may be true or false. Because of this criterion, a semantic principle of bivalence that refers to (actual) truth values of certain propositions is superfluous. The symbolic representation of bipolarity is also a significant difference of Wittgenstein's *ab*-notation with respect to Peirce's α-graphs.

In *ab*-notation, the arbitrary signs a and b are used instead of T and F to emphasise that these poles do not have fixed meanings. Instead, it is only the structure, or, more precisely, the relations of the outermost a- and b-poles to innermost a- and b-poles that is relevant to the meaning of an *ab*-diagram.[16] Thus, all intermediary poles do not contribute to the meaning of an *ab*-diagram; they are not part of the *ab*-symbol but only part of the process to generate the resulting *ab*-symbol. Paraphrasing $a - p - b$ as 'An instance of p is true iff it is true and false iff it is false'[17] makes it explicit (i) that atomic propositions are truth functions of themselves[18] and (ii) that the interpretation of the form of $a - p - b$ arbitrarily decrees that a is to the left of p as a symbol of the condition for truth and that a is not to the right of p as a symbol of the condition for falsehood[19]. By contrast, the symbol p alone does not provide any structural features on which such an interpretation could rely and may, thus, misinterpreted as a name that refers to a truth value.

Sentential connectives, such as $\neg$, $\wedge$, $\vee$ and $\rightarrow$, are translated by *ab*-

Wittgenstein's early writings: CL, letter 32 (p. 57; cf. p. 9); NL, B25, printed in *Biggs 1996*, p. 30; and MN (p. 115).

[15] Cf. CL (letter 28, point (2), p. 47) and NL (p. 102[3]).

[16] Cf. MN (pp. 114[4] and 115[4,5]).

[17] Strictly speaking, this is only a mechanical paraphrase of the *ab*-symbol of p in terms of the two pole-groups $a - \{a - p\}$ and $b - \{b - p\}$; cf. p. 11 below. In contrast to its *ab*-diagram $a - p - b$, in the *ab*-symbol of p, the outermost and innermost poles are separated.

[18] Cf. TLP 5.

[19] Cf. NL (p. 102[4,8]) and MN (p. 115[4,5]))

operations, i.e., by operations that assign *a*- and *b*-poles to *a*- and *b*-poles in turn. Therefore, the formula $\neg p$ is converted into an *ab*-diagram by first writing $a - p - b$ and then applying the *ab*-operation that translates the negator. This operation assigns the *b*-pole to the *a*-pole and the *a*-pole to the *b*-pole. Thus, one derives $b - a - p - b - a$ as the *ab*-diagram of $\neg p$.

In addition to negation, there are 14 other *ab*-operations. These additional operations assign *a*- and *b*-poles to the four pairs of poles *aa*, *ab*, *ba*, and *bb*. Each dyadic sentential connective is defined by one *ab*-operation. Note that *ab*-operations that assign only the *a*-pole or only the *b*-pole to all four pairs of poles do not exist. This fact illustrates a difference between *ab*-operations and truth functions: tautology and contradiction are truth functions but not operations. Sentential connectives are defined as *ab*-operations, not as truth functions, in the *ab*-notation. Their definitions are used to generate *ab*-diagrams in the process of identifying *ab*-symbols. *ab*-symbols alone are unambiguously paraphrased as representations of truth functions in terms of functions of the truth and falsehood of complex propositions of the truth and falsehood of atomic propositions. *ab*-operations can be applied iteratively, and they can cancel each other[20]; they are essential parts of an algorithm. Truth functions, by contrast, result from the interpretation of the result of an algorithmic construction. Identification of a tautology relies not on a special *ab*-operation but rather on a property of an *ab*-diagram that results from applying *ab*-operations. The definitions of the particular sentential connectives are defined with respect to the interpretations of the resulting *ab*-diagrams as truth functions, but they are not identical to the definitions of truth functions. For example, $\vee$ is defined such that the *ab*-diagram of $p \vee q$ represents the truth function 'p or q'. The same, however, is true of $\neg(\neg p \wedge \neg q)$, which does not contain $\vee$. All these differences result from introducing bipolarity as a symbolic feature within the *ab*-notation. Ordinary notation, meanwhile, misleadingly suggests that a formula such as $p \vee q$ should be read in the same way as aRb.

In *ab*-diagrams, *a*- and *b*-poles are grouped in pairs using curly brackets (see Figure 1). Because every dyadic *ab*-operation assigns both the *a*-pole and the *b*-pole to pairs of poles, the four possible pairs of poles, *aa*, *ab*, *ba*, and *bb*, must serve as the bases for any further applications of dyadic *ab*-operations. Therefore, one and the same definition of a given sentential connective applies in every case.

For example, the *ab*-diagram of the formula $p \leftrightarrow p$ is generated by first writing down *apb* twice and then combining the four possible pairs of poles,

[20]Cf. TLP 5.251 and 5.253.

aa, ab, ba, and bb, using curly brackets. Finally, one applies the definition of the sentential connective $\leftrightarrow$ as the operation that assigns the a-pole to aa and bb and the b-pole to ab and ba (see Figure 1).

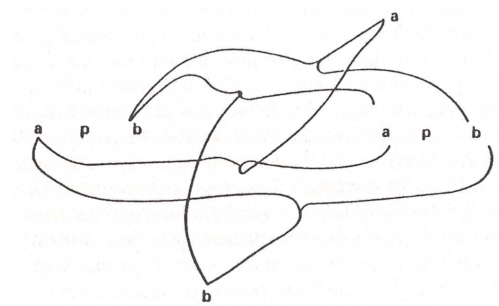


Figure 1: Wittgenstein's ab-diagram of $p \leftrightarrow p$, from CL (p. 57)

The ab-notation implies a crucial structural difference compared with truth tables. In a truth table, 2^n combinations of truth values are considered based on the n propositional variables of different *types* that occur in the propositional formula. By contrast, in an ab-diagram, every combination of poles is considered, regardless of whether the propositional variables are identical. Whereas truth tables are 'type-based', the construction of ab-diagrams is 'token-based'. This allows a pole to be connected to opposite poles of the same propositional variable in an ab-diagram. From this, it follows that tautologies are not identified by the non-existence of an outermost b-pole in an ab-diagram, whereas in truth tables, tautologies are identified by the fact that the truth value F does not occur below the main sentential connective. Instead, ab-diagrams identify tautologies by the fact that the outermost b-pole is assigned to at least one pair of opposite innermost poles connected to two *occurrences* of the *same* propositional variable in any case.[21] This is the symbolic feature of ab-diagrams that is common to all tautologies of propositional logic. It is impossible to interpret tautologies as false because then, a single atomic proposition must be interpreted as both true and false at the same time.

Wittgenstein applies his general identity criterion for tautologies to the above ab-diagram of the formula $p \leftrightarrow p$ in CL, letter 32 (p. 60):

> [...] it is tautological because b is connected only with those pairs of poles that consist of opposite poles of a single proposition (namely p).

[21] Cf. CL, letter 30 (p. 53).

Therefore, the identification of tautologies is traced back to the relations between the outermost *b*-pole and the innermost poles. A 'complex pole of propositional logic' is composed of a propositional variable and one innermost pole, e.g., $a - p$ or $b - p$.[22] Thus, in general, the identification of the truth conditions for formulas is traced back to the relations between the outermost poles and classes of complex poles. Opposite complex poles of propositional logic are complex poles with identical propositional variables but opposite poles. For example, $a-p$ and $b-p$ are opposite complex poles, whereas $a-p$ and $b-q$ are not opposite. The fact that the outermost pole of an *ab*-diagram might be connected to opposite complex poles represents the crucial difference between *ab*-diagrams and truth tables. As will be shown in section 4, this point is crucial for the application of the *ab*-notation to predicate logic. Ignoring this difference, or even maintaining that the rules of the *ab*-notation should be adjusted to comply with the method of truth tables in this respect, as suggested by Black (*1964*, pp. 323-324), makes it impossible to understand the *ab*-notation and its prospects as a notation for FOL.

3.2 *ab*-Symbols

The construction of an *ab*-diagram and the identification of tautologies and contradictions by referring to the relations between the outermost and innermost poles do not suffice to identify equivalent formulas by translating them into one and the same *ab*-symbol in any case. In the following, I will describe how to construct a single *ab*-symbol for all formulas of a class of equivalent propositional formulas. As we will see, this procedure reflects the first step of the Quine-McCluskey algorithm for generating minimised disjunctive normal forms of propositional logic. The results of this procedure are the so-called 'reductive disjunctive normal forms' (RDNFs), which are known to be unique representatives of equivalent classes of propositional logic. To achieve a similar result within the *ab*-notation, two steps are needed: (i) an equivalent to the representation of canonical disjunctive normal forms (CDNFs) and (ii) an equivalent to the first step in the process of minimising CDNFs within the Quine-McCluskey algorithm. As we will see, Wittgenstein's remarks envisage (i) but not (ii).

To represent *ab*-symbols without reproducing insignificant properties of *ab*-diagrams, Wittgenstein describes a method of simplifying *ab*-diagrams

[22] In the notation of complex poles, *a*- as well as *b*-poles are always assigned to the left of propositional variables, cf. p. 11 below.

that results in single ‘pole-groups’ (NL, p. 102[4]), or ‘classes of poles’ (CL, letter 30):

> In place of every proposition p, let us write ${}^{b}_{a}p$. Let every correlation of propositions to each other [...] be effected by a correlation of their poles a and b. Let this correlation be transitive. Then accordingly ${}^{b-b}_{a-a}p$ is the same symbol as ${}^{b}_{a}p$. Let n propositions be given. I then call a ‘class of poles’ of these propositions every class of n members, of which each is a pole of one of the n propositions, so that one member corresponds to each proposition. I then correlate with each class of poles one of two poles (a and b). The sense of the symbolising fact thus constructed I cannot define, but I know it.

This simplified notation abstains from the cumbersome use of curly brackets, and it does not contain any intermediary poles.[23] Instead, it makes use only of innermost poles, which are connected to propositional variables, and of outermost poles, which are connected to pole-groups. I call the notation that assigns outermost poles to classes of complex poles, or pole-groups, the ‘pole-group notation’. Because of the simplicity of the pole-group notation, propositional variables are always provided with poles on their left-hand sides. Therefore, the two ab-diagrams $a - p - b$ and $a - b - a - p - b - a - b$ are represented by the same pole-groups, $a - \{a - p\}$ and $b - \{b - p\}$. In this case, the set of those pole-groups is already the ab-symbol. However, this is not the case in general.

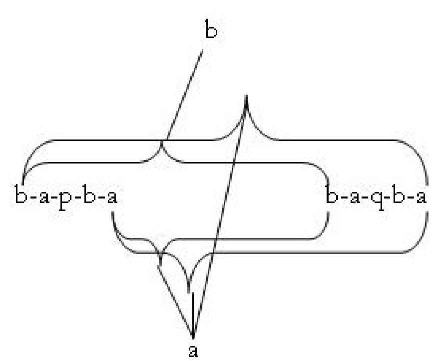


Figure 2: ab-diagram of $\neg p \vee \neg q$

As a first step in moving from ab-diagrams to ab-symbols, ab-diagrams are converted into ab-pole-groups. To do this, each a-pole-group is obtained

[23]Cf. also NL (p. 104[4]), MN (p. 114[7]), and CL (p. 48), point (4), for Wittgenstein’s ‘transitivity rule for poles’, which allows the deletion of all intermediary poles.

as the result for one of the paths from the outermost a-pole to complex poles. The same applies in the case of the b-pole-groups. For example, Figure 2 shows the ab-diagram of the formula

$$\neg p \lor \neg q, \tag{1}$$

which results in the following a- and b-pole groups:

$$\begin{array}{ll} a - \{a - p, & b - q\}, \\ a - \{b - p, & a - q\}, \\ a - \{b - p, & b - q\}, \\ b - \{a - p, & a - q\}. \end{array}$$

Table 1: Pole-groups of $\neg p \lor \neg q$

a-pole-groups represent conditions for truth, whereas b-pole-groups represent conditions for falsehood. a-pole-groups can easily be converted into a disjunctive normal form (DNF) that is equivalent to the initial formula, whereas b-pole-groups are plain representations of a DNF that is equivalent to the negation of the initial formula. In fact, instead of being generated from the rather cumbersome ab-diagrams, pole-groups can be generated directly from an initial formula and its negation through the translation of a rather complex sort of DNF, which I abbreviate 'CCDNF'. For simplicity, we abstain from using any dyadic sentential connectives other than $\land$ and $\lor$ in the following. Then, CCDNFs corresponding to pole-groups can be generated via the following simple procedure:

CCDNF rules:

1. Convert the resulting formula into a negative normal form (NNF) by eliminating double negation and applying De Morgan's laws.

2. Apply the definition of $\lor$, i.e., $A \lor B \dashv\vdash A \land B \lor A \land \neg B \lor \neg A \land B$, to all occurrences of $\lor$.

3. Apply the distributive law $A \lor (B \land C) \dashv\vdash A \land B \lor A \land C$ to obtain a CCDNF.

The CCDNF of formula (1), for example, is

$$p \land \neg q \lor \neg p \land q \lor \neg p \land \neg q, \tag{2}$$

whereas the CCDNF of (¬1) is

$$p \wedge q. \tag{3}$$

These CCDNFs can easily be mapped to the pole-groups of table 1. In general, the resulting CCDNF of an initial formula ϕ can be mapped to the a-pole-groups through a one-to-one mapping of each disjunct of the CCDNF to an a-pole-group. The same applies to the CCDNF of $\neg\phi$ and the b-pole-groups.

To obtain CDNFs from CCDNFs, one must delete all disjuncts that contain both a propositional variable A that is not negated and its negation $\neg A$. If no disjunct remains after this deletion, then the initial formula is a contradiction, and consequently, its negation is a tautology. This corresponds to Wittgenstein's rule for identifying tautologies based on ab-diagrams in relation to the connections between the outermost poles and opposite complex poles. I call pole-groups that contain opposite complex poles 'contrary pole-groups'. When an initial formula is identified as a tautology or contradiction, the construction of the ab-symbol immediately terminates. In this case, the ab-symbol consists of empty pole-groups for one pole and 'total' pole-groups for the other. In accordance with Wittgenstein's metaphor by which a tautology 'leav[es] the whole of logical space open to reality', whereas a contradiction 'leaves no point of it to reality' (TLP 4.463), one might symbolise tautologies by $a - \{\square\}$ and $b - \{\blacksquare\}$ and contradictions by the opposite. This symbolisation shows that tautologies and contradictions do not depend on any specific proposition.

To obtain CDNFs and their respective pole-groups in the case that the initial formula is neither a tautology nor a contradiction, one must ultimately delete identical disjuncts (pole-groups) and identical conjuncts (complex poles), leaving only one occurrence in each case. Let us call the pole-groups corresponding to CDNFs 'canonical pole-groups'. The isomorphism of Wittgenstein's ab-notation, and pole-groups in particular, to DNFs is a further crucial difference with respect to Peirce's α-graphs, which are isomorphic to formulas that contain only $\neg$ and $\wedge$.

The translation procedure into canonical pole-groups described thus far suffices to assign to formulas (1) and (2) the same pole-groups listed in table 1. However, consider the case of p and $p \wedge q \vee p \wedge \neg q$: These two formulas are also equivalent, but their CDNFs and the corresponding canonical pole-groups are not identical:

$$a - \{a - p\},$$
$$b - \{b - p\}.$$

Table 2: Canonical pole-groups of p

$$a - \{a - p, a - q\},$$
$$a - \{a - p, b - q\},$$
$$b - \{b - p, a - q\},$$
$$b - \{b - p, b - q\}.$$

Table 3: Canonical pole-groups of $p \wedge q \vee p \wedge \neg q$

Wittgenstein's remarks on the *ab*-notation do not offer any hints regarding how to proceed in this case. However, the aim of Wittgenstein's iconic proof procedure can easily be achieved if one implements the so-called merging process of the Quine-McCluskey algorithm to obtain RDNFs from CDNFs. This process is based on the following so-called 'merging rule':

$$A_1 \wedge \ldots \wedge A_n \wedge B \vee A_1 \wedge \ldots \wedge A_n \wedge \neg B \quad \dashv\vdash \quad A_1 \wedge \ldots \wedge A_n \quad \text{(M-R).}$$

In the Quine-McCluskey algorithm, disjuncts are represented as lists of sets, just like pole-groups. Thus, the algorithm is directly applicable to pole-groups with identical outermost poles. The algorithm iteratively applies M-R to sets of identical length. If M-R is applicable, then the new term is placed in a new list. Pairs of compared terms must be marked, but unpaired terms can be used for further applications of M-R until this rule can no longer be further applied to the terms in a list. Identical terms must be written down only once in the new list. All terms that are not marked to indicate that M-R is no longer applicable to them are the elements of the resulting RDNF or, correspondingly, the resulting reductive pole-groups. When applied to the pole-groups in table 3, this procedure results in the list given in table 2.

The Quine-McCluskey algorithm goes on to minimise RDNFs. However, as is well known, this algorithm does not result in unique solutions. The RDNF given in (4) below, for example, can be reduced to either (5) or (6), which are both equivalent to and shorter than (4):

$$P \wedge \neg Q \vee \neg P \wedge Q \vee P \wedge R \vee Q \wedge R, \tag{4}$$
$$P \wedge \neg Q \vee \neg P \wedge Q \vee P \wedge R, \tag{5}$$
$$P \wedge \neg Q \vee \neg P \wedge Q \vee Q \wedge R. \tag{6}$$

Thus, this minimisation step does not constitute part of an iconic proof procedure. However, an algorithm that generates CDNFs (or CCDNFs and then CDNFs) from a given propositional formula and then reduces the CDNFs to RDNFs satisfies the claim that all formulas of a class of equivalent propositional formulas can be translated into one and the same RDNF. The same applies to an algorithm that starts by generating *ab*-diagrams and then converts those diagrams into canonical pole-groups and, finally, into reductive pole-groups. The resulting reductive pole-groups are the *ab*-symbols of the initial formulas. The conditions for truth and falsehood can be directly read off from the *ab*-symbols by translating *a*-pole-groups as conditions for truth and *b*-pole-groups as conditions for falsehood.

4 Elementary FOL

This section describes how to construct a single *ab*-symbol for all formulas of a class of equivalent elementary FOL formulas.

4.1 *ab*-Diagrams

There is only one passage in Wittgenstein's known early writings that concerns the *ab*-notation of predicate logic, from NL (pp. 95f.):[24]

> The application of the *ab*-notation to apparent–variable propositions becomes clear if we consider that, for instance, the proposition 'for all x, φx' is to be true when φx is true for all x's and false when φx is false for some x's. We see that *some* and *all* occur simultaneously in the proper apparent variable notation.
>
> The notation is:
> for $\forall x \varphi x : a - \forall x - a\varphi x b - \exists x - b$ and
> for $\exists x \varphi x : a - \exists x - a\varphi x b - \forall x - b$
>
> Old definitions now become tautologous.

Wittgenstein derives his *ab*-notation for the two most primitive quantifier-containing formulas from a standardised paraphrasing of the truth and falsehood conditions of their instances. The *ab*-notation is designed to identify the truth and falsehood conditions of propositions based on symbolic properties. Therefore, it turns out that both quantifiers constitute an irreducible part of a proper notation. This constitutes a significant difference with respect to truth tables and to any attempt to reduce FOL to propositional logic. It is also a significant difference with respect to Peirce's β-graphs.

[24] I tacitly replace Wittgenstein's Russellian notation for quantifiers with modern notation; i.e., I write $\forall x$ instead of (x) and $\exists x$ instead of $(\exists x)$.

The purpose of the *ab*-notation is to identify the logical properties of *any* formula of a certain realm based on symbolic criteria. Axioms and definitions (or equivalence rules) must therefore be identified as tautologies in the *ab*-notation. This is Wittgenstein's meaning in the last sentence of the quotation. By the term 'old definitions', he refers to quantifier definitions such as those stated in propositions $*9.01 - 9.08$ of *Principia Mathematica*. These definitions define expressions with sentential connectives outside the scope of quantifiers by means of prenex normal forms.[25] The first definition is as follows:

$$*9.01 \;\; \neg\forall x\varphi x = \exists x\neg\varphi x \; Df.$$

Wittgenstein claims that his *ab*-notation makes it possible to prove such a definition. Therefore, any general rule for identifying tautologies in the *ab*-notation must apply to $\neg\forall x\varphi x \leftrightarrow \exists x\neg\varphi x$, and any transformation of $\neg\forall x\varphi x$ and $\exists x\neg\varphi x$ must result in identical *ab*-symbols.

To construct *ab*-diagrams of elementary FOL, one must translate an FOL formula ϕ from inside to outside in accordance with the logical hierarchy of ϕ. In elementary FOL, quantifiers must be considered in addition to sentential connectives.

***ab*-Diagrams:**

1. Provide tokens of the propositional functions of ϕ with an innermost a-pole to the left and an innermost b-pole to the right, e.g., $A \Rightarrow a{-}A{-}b$.

2. (a) Assign the quantifier of the formula ϕ plus its variable to the a-pole of the appropriate part of the *ab*-diagram under construction.

   (b) Assign the a-pole to this quantifier.

   (c) Proceed similarly for the b-pole of the appropriate part of the *ab*-diagram by assigning to it the opposite quantifier, supplemented

[25] *Russell 1910*, p. 136 (here, as in the quotation from *9.01, I tacitly replace the Russellian notation for quantifiers with modern notation):

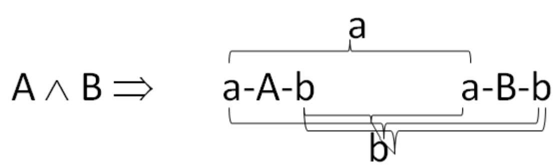
> In virtue of these definitions, the true scope of an apparent variable is always the whole of the asserted proposition in which it occurs, even when, typographically, its scope appears to be only part of the asserted proposition. Thus when $\exists x\phi x$ or $\forall x\phi x$ *appears* as *part* of an asserted proposition, it does not really occur, since the scope of the apparent variable really extends to the whole asserted proposition.

By contrast, Wittgenstein's conception of logic gives priority to anti-prenex normal forms; cf. *Lampert 2017*.

with an outward b-pole, e.g.,

$$\forall x A(x) \Rightarrow a - \forall x - a - A(x) - b - \exists x - b$$

3. Translate the sentential connectives in accordance with their definitions as ab-operations (see p. 8 above), e.g.,



For an example implying dyadic sentential connectives and, consequently, curly brackets, see Figure 4 on p. 28.

Applying the above rule results in, e.g., Wittgenstein's previously quoted ab-diagrams for $\forall x \varphi x$ and $\exists x \varphi x$. In the cases of $\neg \forall x \varphi x$ and $\exists x \neg \varphi x$, one obtains $b-a-\forall x-a-\varphi x-b-\exists x-b-a$ and $b-\forall x-b-a-\varphi x-b-a-\exists x-a$. These two diagrams are identical symbols, which becomes clear from the fact that the positions of neither the quantifiers nor the intermediary poles are part of the ab-symbol. All that matters is the relations of the outward poles and the quantifiers to the inward poles of the propositional functions. In this respect, the two ab-diagrams do not differ from each other.

To apply Wittgenstein's rule for identifying tautologies, one must generalise the notion of 'opposite complex poles' (see p. 10 above). Clearly, it is not sufficient to refer to opposite (single) poles of identical propositional functions because $\exists x Fx$ and $\exists x \neg Fx$ are not contradictory. Instead, quantifiers must be part of complex poles. From this, it follows that the internal relations between complex poles in elementary FOL must be identified in addition to the trivial internal relations of identity and contradiction that are considered in propositional logic, e.g., $a - p$ being identical to $a - p$ and $a - p$ being contradictory to $b - p$. Therefore, the identification of the logical properties of formulas in elementary FOL essentially reduces to the identification of the internal relations between complex poles involving quantifiers. This is why the rules for constructing ab-diagrams must not involve internal relations between complex poles, as would be the case if those rules were type-based (instead of token-based, see p. 9 above), as they are in the case of truth tables or CDNFs. I will elucidate this point in the next section by explaining how to identify ab-symbols given ab-diagrams in elementary FOL; see section 4.2.2 in particular.

4.2 *ab*-Symbols

4.2.1 Pole-groups

To specify how to convert *ab*-diagrams into pole-groups in elementary FOL, let us define a 'path' as a concatenation of poles and quantifiers leading, via curly brackets and intermediate poles, from an outermost pole of the *ab*-diagram to the innermost poles of the propositional functions. These paths branch if a pole is connected to pairs of poles by *several* brackets. Each path runs over a single such branch. Quantifiers occur only inside brackets in the *ab*-diagrams of elementary FOL. Therefore, the paths no longer branch after having reached quantifiers. In contrast to the complex poles of propositional logic, the complex poles of elementary FOL contain quantifiers in addition to innermost poles and propositional functions. Each pole-group is generated from a single path leading from an outermost pole to complex poles. The *a*-pole-groups are obtained from the paths starting at the outermost *a*-pole; the *b*-pole-groups are obtained from the paths starting at the outermost *b*-pole. All intermediary poles along these paths are to be eliminated, after which the outermost pole plus the complex poles will remain for each path. A path without its intermediary poles is called an 'elementary path'. A pole-group corresponds to each elementary path. This procedure is based on Wittgenstein's rule of transitivity (see p. 11 above).

Pole-groups: To convert an *ab*-diagram into pole-groups, generate all elementary paths. Each elementary path starting with the *a*-pole corresponds to an *a*-pole-group; each elementary path starting with the *b*-pole corresponds to a *b*-pole-group.

Converting *ab*-diagrams into pole-groups leads to identical pole-groups in the case of the two *ab*-diagrams $b - a - \forall x - a - \varphi x - b - \exists x - b - a$ and $b - \forall x - b - a - \varphi x - b - a - \exists x - a$ (see p. 16 above):

$$a - \{\exists x - b - \varphi x\},$$
$$b - \{\forall x - a - \varphi x\}.$$

However, not all equivalent formulas are assigned to identical pole-groups. In moving from pole-groups to *ab*-symbols, the first step concerns the single complex poles of elementary FOL ('elementary complex poles' for brevity). In contrast to propositional logic, equivalent but not identical elementary complex poles exist in elementary FOL, e.g., $\exists x - a - Fx$ and $\exists y - a - Fy$. These equivalences are due to the renaming of variables (variable definitions). To identify these equivalences from their identical properties in the *ab*-notation, one must consider that the specific types of the variables do not contribute to the meaning of a quantified proposition. Instead, 'they serve merely to indicate cross-references to various positions of quantification' (*Quine*, (*1983*, p. 70)). To indicate this, I use a notation similar to Peirce's 'lines of identity' or Quine's 'bonds' that connect positions of propositional functions that are assigned to quantifiers. Thus, e.g., the complex poles $\exists x - a - Fxx$ and $\exists y - a - Fyy$ are both translated into $\exists <^{1}_{2}\, a - F_{12}$. For convenience, I do not directly relate lines to positions as Peirce and Quine do; rather, I indicate positions by numbers and connect those numbers with lines. I call lines that connect positions 'forks'. I call the resulting complex poles 'symbolising complex poles', or 'complex poles' or 'symbolising poles' for brevity. Each is composed of a 'prefix' (i.e., a sequence of names and quantifiers + forks connecting numbers) and a 'suffix' (i.e., an innermost pole + a propositional function). Similar to Wittgenstein's suggestion in an NB entry from December 2nd, 1916, names are treated like quantifiers and head the prefix. Unlike for variables, the specific types of these names matter. Hence, positions are connected to names as they are to quantifiers. Symbolising poles are generated from elementary poles in the following way:

Symbolising complex poles:

1. Replace the variables that are bound by quantifiers with the numbers representing their positions in the propositional function. If a variable occurs more than once in the propositional function, then connect the corresponding numbers with a 'fork', e.g. translate the bound variable x into $<^{1}_{2}$.

2. If names occur, then prefix them to the complex pole in alphabetical order and assign to each name the number corresponding to its position in the propositional function. If a name occurs more than once in the propositional function, then connect the corresponding numbers with a fork, e.g., translate $a - Fcc$ into $c <^{1}_{2} a - F_{12}$.

3. Replace each variable and name in the propositional function with the number corresponding to its position in the propositional function, e.g., replace $Fxxy$ with F_{123}.

4. To indicate that the order of the names, existential quantifiers, or universal quantifiers in each corresponding sequence thereof is insignificant, all names, existential quantifiers, and universal quantifiers are

separated by commas within those sequences. For the same reason, the preceding sequence of names is separated by a comma from the following sequence of quantifiers.

Therefore, the elementary complex pole

$$\exists x \forall y - a - Fxxyccd$$

is written as

$$c <^4_5, d_6, \exists <^1_2 \forall_3 - a - F_{123456},$$

which is paraphrased as follows: 'The object c in the fourth position and the fifth position combined with the object d in the sixth position combined with some object, the same in the first position and the second position, combined with all objects in the third position makes the 6-adic function F true'. For another example that illustrates how to yield pole-groups with symbolising poles from the *ab*-diagram depicted in Figure 4, see p. 29.

To generate *ab*-symbols from pole-groups that contain symbolising complex poles, only those complex poles that contribute to an unambiguous representation of the conditions for truth and falsehood within the specified groups of symbolising poles must be identified. All other complex poles, although they are 'symbolising' if paraphrased on their own, do not 'symbolise' in the pole-groups in which they occur; they are not part of the *ab*-symbol because they are not necessary for unambiguously identifying the conditions for truth and falsehood. For example, it is obvious that both of two identical symbolising complex poles do not symbolise within a single pole-group. In general, an unambiguous minimisation procedure is needed to eliminate any symbolising complex poles that do not symbolise in the specific contexts of their occurrence in pole-groups. In propositional logic, this is essentially the counterpart to the first step of the Quine-McCluskey algorithm for deriving RDNFs. Before defining a similar procedure for elementary FOL, a general rule for identifying the internal relations between complex poles of elementary FOL must be defined.

4.2.2 Internal Relations between Complex Poles

In propositional logic, the only internal relation that can exist between two different complex poles is a relation of contradiction. This relation can be trivially identified: The same propositional variable is prefixed by 'opposite' poles. In the case of symbolising complex poles, contradictory complex poles are identified in a similar but slightly generalised way. Because the prefix also contains quantifiers, one must exchange existential and universal quantifiers to generate the corresponding ‘opposite prefix’. Therefore, e.g., the opposite (contradictory) complex pole of $c <^4_5, d_6, \exists <^1_2 \forall_3 - a - F_{123456}$ is $c <^4_5, d_6, \forall <^1_2 \exists_3 - b - F_{123456}$.

However, in addition to contradiction relations, relations of implication as well as subcontrary and contrary relations can exist between elementary complex poles. Subcontrary and contrary relations can be defined in terms of implication and contradiction: A and B are subcontrary iff the contradiction of A implies B; A and B are contrary iff B implies the contradiction of A. Thus, it is sufficient to determine only the relations of implication in addition to the relation of contradiction. In elementary FOL, this task can be reduced to defining the relations between the prefixes of complex poles because poles with different suffixes cannot be related by logical implication. Table 4 defines a correct and complete calculus for identifying the relations of implication between elementary complex poles with identical suffixes. μ and ν represent forks connecting numbers; as a limiting case a fork may have one peak only. $<^<$ is identical to $\leqslant$ a.s.f. for more complex iterations of forks.

$\exists\forall Ex$:	$\exists\mu\forall\nu \vdash \forall\nu\exists\mu$		
$\forall E$:	$\forall\mu \vdash t\mu$	$\exists I$:	$t <^{\mu}_{\nu} \vdash t\mu, \exists\nu$
$< I1$:	$\forall\mu, \forall\nu \vdash \forall <^{\mu}_{\nu}$	$< E1$:	$\exists <^{\mu}_{\nu} \vdash \exists\mu, \exists\nu$
$< I2$:	$\exists\mu\forall\nu \vdash \exists <^{\mu}_{\nu}$	$< E2$:	$\forall <^{\mu}_{\nu} \vdash \forall\mu\exists\nu$

Table 4: Rules of implication

The rules of table 4 specify only minimal differences between otherwise identical elementary complex poles. Their correctness can easily be verified by translating these rules into ordinary notation. Alternatively, their paraphrasing is clear and may suffice. Their completeness can be verified by enumerating all remaining possible minimal symbolic variations and arguing, using either a paraphrase or model theory, that the corresponding transitions are not truth-preserving. A calculus based on ‘minimal syntactic differences’ between symbols of a proper notation that allows one to decide upon relations of implication may be regarded as a typical feature of an iconic proof procedure. Table 5 lists a set of rules that is sufficient to constitute the minimal symbolic variations to rule out relations of implication (= $\overline{\text{rules of implication}}$). s and t are variables of different names.

$\overline{\varphi/\psi}$:	$\phi \not\vdash \psi$	$\overline{a/b}$:	$a - \varphi \not\vdash b - \varphi$ $b - \varphi \not\vdash a - \varphi$
$\overline{\forall\exists Ex}$:	$\forall\mu\exists\nu \not\vdash \exists\nu\forall\mu$		
$\overline{tI}$:	$s\mu \not\vdash t\mu$	$\overline{\exists E}$:	$\exists\mu \not\vdash t\mu$
$\overline{< E1}$:	$\forall <^{\mu}_{\nu} \not\vdash s\mu, t\nu$	$\overline{< I1}$:	$s\mu, t\nu \not\vdash \exists <^{\mu}_{\nu}$
$\overline{< E2}$:	$\forall <^{\mu}_{\nu} \not\vdash \exists\mu\forall\nu$	$\overline{< I2}$:	$\forall\mu\exists\nu \not\vdash \exists <^{\mu}_{\nu}$

Table 5: $\overline{\text{Rules of implication}}$

There is no need for, e.g., a $\overline{\text{rule of implication}}$ that states that $\exists\mu \not\vdash \forall\mu$ because of the following principle: If the transition from a symbolising property X to a symbolising property Y is not justified, then the transition from X to a symbolising property Y^*, from which Y can be derived by applying one of the rules of implication, is, a fortiori, also not justified. I call this principle the 'strengthening of the consequent' (SC). In a similar manner, an additional principle of the 'weakening of the antecedent' (WA) must be presumed for a complete classification of minimal symbolic differences and their distinction in truth-preserving and non-truth-preserving transitions.

Whether a relation of implication exists between two complex poles can be determined either by constructing complete implication trees (see Figure 3) or by means of an efficient decision rule that prescribes how to pass from one pole to another using the rules of implication, if possible, or a $\overline{\text{rule of implication}}$, if necessary (see Example 1 and Example 2 below). These scanty remarks in combination with their exemplification must suffice in this outline of Wittgenstein's *ab*-notation.

Example 1:

$$t_1, \exists <^{8}_{9} \forall_4, \forall_5 \exists <^{6}_{7} \forall <^{2}_{3} - a - F_{123456789}$$
$$\vdash$$
$$t_3, \forall <^{4}_{5} \exists <^{1}_{2}, \exists_6, \exists_7, \exists_8, \exists_9 - a - F_{123456789}$$

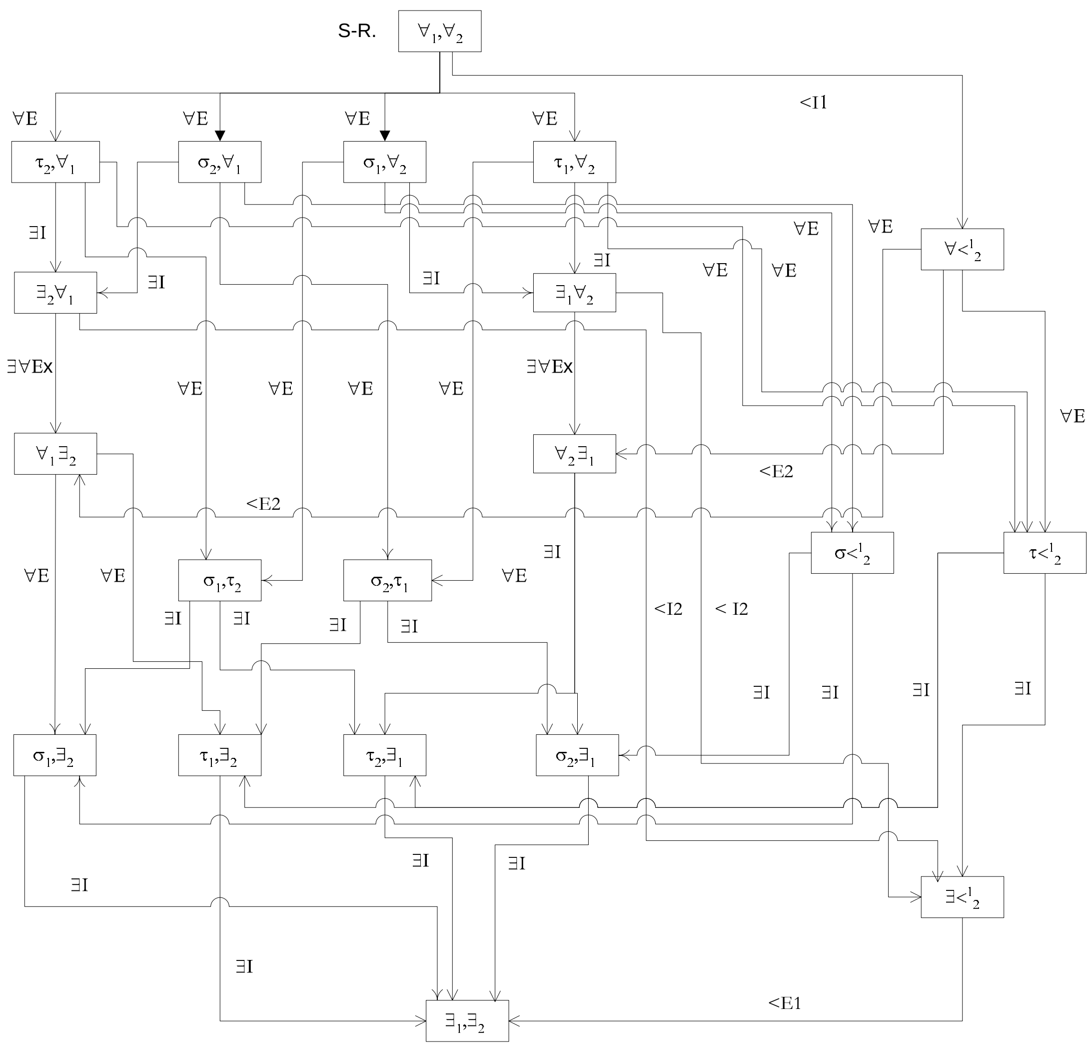


Figure 3: Implication tree for prefixes of length 2 with 2 names

No.	Pole	Rule
1.	$t_1, \exists < {}^{8}_{9}\forall_4, \forall_5\exists < {}^{6}_{7}\forall < {}^{2}_{3} - a - F_{123456789}$	A
2.	$t_1, \exists < {}^{8}_{9}\forall < {}^{4}_{5}\exists < {}^{6}_{7}\forall < {}^{2}_{3} - a - F_{123456789}$	$< I1$
3.	$t \lessdot_{\substack{1\\2\\3}}, \exists < {}^{8}_{9}\forall < {}^{4}_{5}\exists < {}^{6}_{7} - a - F_{123456789}$	$\forall E$
4.	$t \lessdot_{\substack{1\\2\\3}}, \exists < {}^{8}_{9}\forall < {}^{4}_{5}\exists_6, \exists_7 - a - F_{123456789}$	$< E1$
5.	$t \lessdot_{\substack{1\\2\\3}}, \exists_8, \exists_9\forall < {}^{4}_{5}\exists_6, \exists_7 - a - F_{123456789}$	$< E1$
6.	$t_3, \exists < {}^{1}_{2}, \exists_8, \exists_9\forall < {}^{4}_{5}\exists_6, \exists_7 - a - F_{123456789}$	$\exists I$
7.	$t_3, \forall < {}^{4}_{5}\exists < {}^{1}_{2}, \exists_6, \exists_7, \exists_8, \exists_9 - a - F_{123456789}$	$\exists\forall Ex$

EXAMPLE 2:

$$\exists_4\forall_5\exists_3\forall < {}^{1}_{2} - b - F_{12345} \quad \not\vdash \quad \forall_1\exists_4\forall_5\exists < {}^{2}_{3} - b - F_{12345}$$

No.	Pole	Rule
1.	$\exists_4\forall_5\exists_3\forall < {}^{1}_{2} - b - F_{12345}$	A
2.	$\exists_4\forall_5\exists \lessdot_{\substack{1\\2\\3}} - b - F_{12345}$	$< I2$
3.	$t_1, \exists_4\forall_5\exists < {}^{2}_{3} - b - F_{12345}$	$\overline{\exists E}$

The proof terminates on line 3 with the first necessary application of a rule of implication. This application is necessary because to proceed from the complex pole in line 2 to $\forall_1\exists_4\forall_5\exists < {}^{2}_{3} - b - F_{12345}$, a transition from $\exists_1$ to $\forall_1$ is necessary. However, such a transition is invalid because of $\overline{\exists E}$ and (SC).

4.2.3 Minimisation

Given the pole-groups with symbolising poles (= symbolising pole-groups), the next step in the process of generating the *ab*-symbol is to generate 'canonical' symbolising pole-groups. This is a first stage of minimisation similar to the generation of canonical pole-groups within propositional logic (see p. 13 above). The only difference with respect to propositional logic is that a 'contrary pole-group' is now defined more generally as a pole-group that contains at least two *contrary* complex poles. According to the definitions given in the previous section, whether two complex poles are contrary is decidable. Thus, canonical pole-groups within elementary FOL can be obtained by applying the following rules:

Canonical symbolising pole-groups:

1. Eliminate all contrary pole-groups of complex poles.
2. List each pole of a pole-group only once.
3. List each pole-group only once.

The application of these rules is illustrated on p. 29 to 31. These rules mirror the procedure for generating a CDNF from a CCDNF in propositional logic. CCDNFs within elementary FOL are generated by the CCDNF rules defined on p. 12, of which the first rule must additionally make use of quantifier definitions to achieve NNFs within elementary FOL.

The first rule for generating canonical symbolising pole-groups already suffices to determine whether an initial formula of elementary FOL is a tautology: it is a tautology iff no *b*-pole-group remains after rule 1 is applied. Similarly, the initial formula is a contradiction iff no *a*-pole-group remains. This is so because each single pole-group corresponds to a disjunct of a corresponding DNF of elementary FOL, which is contradictory iff each disjunct contains at least two conjuncts that are contrary.

The second stage in the minimisation process for generating *ab*-symbols from pole-groups is a generalisation of the merging rule applied in the first step of the Quine-McCluskey algorithm. This merging rule can be generalised as follows to consider the internal relations of complex poles of elementary FOL:

MR: If a pole-group PG1 contains a complex pole $A1$, another pole-group PG2 with the same outermost pole contains a complex pole $A2$, $A1$ and $A2$ are subcontrary, and all other complex poles of PG2 are implied by some complex pole of PG1, then eliminate $A1$ from PG1.

This can be justified by (i) translating the rule into ordinary notation, (ii) converting the corresponding disjunction into a conjunctive normal form (CNF), (iii) minimising this CNF expression by applying tautology elimination and the counterparts to rules IR1 and IR2 below, (iv) converting the result back into a disjunction of conjunctions, and (v) minimising the resulting DNF by again applying the counterparts to IR1 and IR2.

Unlike in propositional logic, complex poles of elementary FOL may be related through logical implication. Therefore, two further rules, in addition to MR, are needed to generate unambiguous representatives of classes of equivalent elementary FOL formulas.

Rule of implication 1 (IR1): If a pole-group PG contains two complex poles, $A1$ and $A2$, such that $A1$ implies $A2$, eliminate $A2$.

Rule of implication 2 (IR2): If each complex pole of a pole-group PG1 is implied by some complex pole of another pole-group PG2 with the same outermost pole, eliminate PG2.

Again, these rules are evident if one considers their translations into ordinary notation.

To guarantee that MR is applied to the maximal extent, (i) the process of merging must precede the process of minimising the pole-groups in accordance with IR1 and (ii) no pole-groups can be eliminated during merging. This ensures the availability of the maximum number of pole-groups that can induce the minimisation of a pole-group due to MR. To avoid repeated application of MR to the same pair of pole-groups, the pairs of pole-groups to which MR has already been applied must be marked. This is also similar to the Quine-McCluskey algorithm. These considerations result in the following rules for stage 2 of the minimisation process starting from canonical pole-groups:

***ab*-symbol:**

1. Apply IR2 to the maximal extent.
2. If MR is applicable, apply MR to the maximal extent to add the new minimised pole-groups to the existing pole-groups and return to step 1; otherwise, go to step 3.
3. Apply IR1 to the maximal extent. Finally, apply IR2 to the result once more.

The resulting pole-groups constitute the *ab*-symbol.

As it does in the case of RDNFs in propositional logic, this procedure leads to unambiguous representations for all formulas of a class of equivalent formulas in elementary FOL. This is evident from the fact that the result represents the *maximum* of the different *minimal* conditions for truth and falsehood for instances of the initial formula.

In this sense, the resulting *ab*-symbol identifies the logical form of the initial formula. This form can be directly read from the *ab*-symbol by paraphrasing the symbolising properties of the *ab*-symbol in accordance with a mechanical reading algorithm.

Reading Algorithm: Paraphrase the resulting *ab*-symbol of a formula ϕ from the outside to the inside according to the following rules:

1. Start by paraphrasing the a-pole-groups using the phrase ‘An instance of the formula ϕ is true iff’ and the b-pole-groups using the phrase ‘An instance of the formula ϕ is false iff’. Connect the paraphrases of the a- and b-pole-groups with ‘and’.

2. Connect the paraphrases of pole-groups with identical outmost pole with ‘or’ and the paraphrases of complex poles with ‘and’.

3. Paraphrase the complex poles from the outside to the inside as follows:

   (a) $\exists$ is paraphrased as ‘some object’, $\forall$ is paraphrased as ‘all objects’, and a name t is paraphrased as ‘the object t’.

   (b) Each fork that succeeds a quantifier is paraphrased as ‘the same’.

   (c) Each number k subsequent to a fork is paraphrased as ‘in the kth position’, and these paraphrases are connected with ‘and’.

   (d) Connect the paraphrases of names, quantifiers plus subsequent forks and numbers with ‘combined with’.

   (e) Paraphrase $a - A_{1...n}$ as ‘makes the n-adic propositional function A true’ and $b - A_{1...n}$ as ‘makes the n-adic propositional function A false’. In the case of a propositional variable, paraphrase $a - A$ as ‘A is true’ and $b - A$ as ‘A is false’.

For an example of the paraphrase of the complex pole $c <_5^4, d_6, \exists <_2^1 \forall_3 - a - F_{123456}$, see p. 20 above, and for the paraphrase of an ab-symbol, see p. 32 below.

These rules show that the conditions for truth and falsehood in elementary FOL are functions of complex poles. These complex poles are, in a sense, the atomic constituents of elementary FOL. The resulting ab-symbols also make it possible to identify logical relations between formulas by applying the calculus to identify internal relations between symbolising complex poles (see section 4.2.2) in combination with simple rules regarding the addition and elimination of complex poles and pole-groups. Thus, the procedure satisfies the aim of serving as an iconic proof procedure that translates formulas expressed in a deficient notation into ideal symbols that allow one to identify logical properties and relations from the properties of those symbols.

5 Example

The following example illustrates the process of generating the *ab*-symbol of an ordinary formula.

We start with the following formula:

$$\forall z \exists y \forall x Fxyz \wedge \forall x \exists y Fxyx \wedge \forall x \forall y Fxxy \vee \exists x Fxxx. \tag{7}$$

Figure 4 presents the *ab*-diagram.

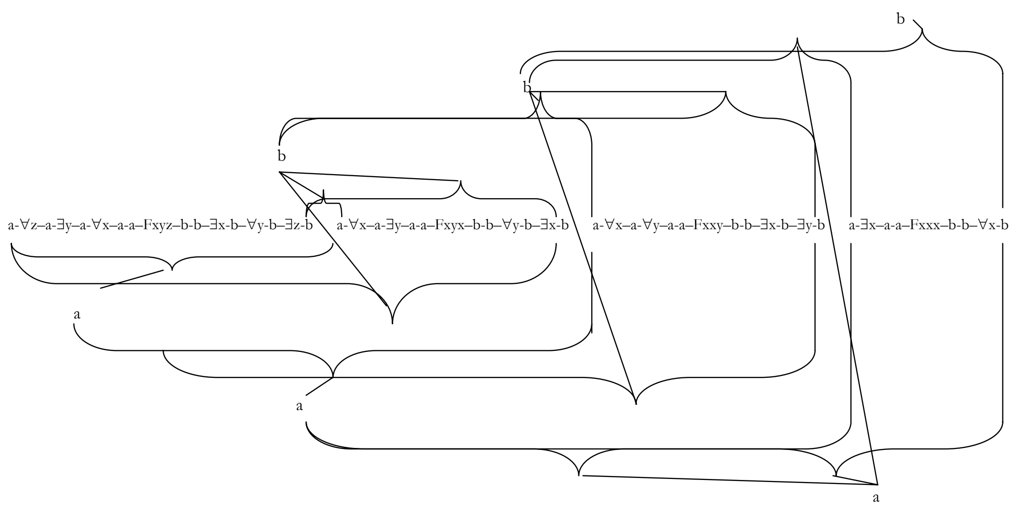


Figure 4: *ab*-diagram

The *ab*-diagram in Figure 4 can be translated into the following symbolising pole-groups:

1. $a - \{\forall_3 \exists_2 \forall_1 - a - F_{123}, \quad \forall < \substack{1\\3} \exists_2 - a - F_{123}, \quad \forall < \substack{1\\2}, \forall_3 - a - F_{123}, \quad \exists <\substack{1\\2\\3} - a - F_{123}\},$
2. $a - \{\forall_3 \exists_2 \forall_1 - a - F_{123}, \quad \forall < \substack{1\\3} \exists_2 - a - F_{123}, \quad \forall < \substack{1\\2}, \forall_3 - a - F_{123}, \quad \forall <\substack{1\\2\\3} - b - F_{123}\},$
3. $a - \{\forall_3 \exists_2 \forall_1 - a - F_{123}, \quad \forall < \substack{1\\3} \exists_2 - a - F_{123}, \quad \exists < \substack{1\\2}, \exists_3 - b - F_{123}, \quad \exists <\substack{1\\2\\3} - a - F_{123}\},$
4. $a - \{\forall_3 \exists_2 \forall_1 - a - F_{123}, \quad \exists < \substack{1\\3} \forall_2 - b - F_{123}, \quad \forall < \substack{1\\2}, \forall_3 - a - F_{123}, \quad \exists <\substack{1\\2\\3} - a - F_{123}\},$
5. $a - \{\exists_3 \forall_2 \exists_1 - b - F_{123}, \quad \forall < \substack{1\\3} \exists_2 - a - F_{123}, \quad \forall < \substack{1\\2}, \forall_3 - a - F_{123}, \quad \exists <\substack{1\\2\\3} - a - F_{123}\},$
6. $b - \{\forall_3 \exists_2 \forall_1 - a - F_{123}, \quad \forall < \substack{1\\3} \exists_2 - a - F_{123}, \quad \exists < \substack{1\\2}, \exists_3 - b - F_{123}, \quad \forall <\substack{1\\2\\3} - b - F_{123}\},$

7.	$b - \{\forall_3 \exists_2 \forall_1 - a - F_{123},$	$\exists <^{1}_{3} \forall_2 - b - F_{123},$	$\forall <^{1}_{2}, \forall_3 - a - F_{123},$	$\forall <^{1}_{\substack{2\\3}} - b - F_{123}\},$
8.	$a - \{\forall_3 \exists_2 \forall_1 - a - F_{123},$	$\exists <^{1}_{3} \forall_2 - b - F_{123},$	$\exists <^{1}_{2}, \exists_3 - b - F_{123},$	$\exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$
9.	$b - \{\exists_3 \forall_2 \exists_1 - b - F_{123},$	$\forall <^{1}_{3} \exists_2 - a - F_{123},$	$\forall <^{1}_{2}, \forall_3 - a - F_{123},$	$\forall <^{1}_{\substack{2\\3}} - b - F_{123}\},$
10.	$a - \{\exists_3 \forall_2 \exists_1 - b - F_{123},$	$\forall <^{1}_{3} \exists_2 - a - F_{123},$	$\exists <^{1}_{2}, \exists_3 - b - F_{123},$	$\exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$
11.	$a - \{\exists_3 \forall_2 \exists_1 - b - F_{123},$	$\exists <^{1}_{3} \forall_2 - b - F_{123},$	$\forall <^{1}_{2}, \forall_3 - a - F_{123},$	$\exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$
12.	$b - \{\forall_3 \exists_2 \forall_1 - a - F_{123},$	$\exists <^{1}_{3} \forall_2 - b - F_{123},$	$\exists <^{1}_{2}, \exists_3 - b - F_{123},$	$\forall <^{1}_{\substack{2\\3}} - b - F_{123}\},$
13.	$b - \{\exists_3 \forall_2 \exists_1 - b - F_{123},$	$\forall <^{1}_{3} \exists_2 - a - F_{123},$	$\exists <^{1}_{2}, \exists_3 - b - F_{123},$	$\forall <^{1}_{\substack{2\\3}} - b - F_{123}\},$
14.	$b - \{\exists_3 \forall_2 \exists_1 - b - F_{123},$	$\exists <^{1}_{3} \forall_2 - b - F_{123},$	$\forall <^{1}_{2}, \forall_3 - a - F_{123},$	$\forall <^{1}_{\substack{2\\3}} - b - F_{123}\},$
15.	$a - \{\exists_3 \forall_2 \exists_1 - b - F_{123},$	$\exists <^{1}_{3} \forall_2 - b - F_{123},$	$\exists <^{1}_{2}, \exists_3 - b - F_{123},$	$\exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$
16.	$b - \{\exists_3 \forall_2 \exists_1 - b - F_{123},$	$\exists <^{1}_{3} \forall_2 - b - F_{123},$	$\exists <^{1}_{2}, \exists_3 - b - F_{123},$	$\forall <^{1}_{\substack{2\\3}} - b - F_{123}\}.$

Table 8: Symbolising complex poles of formula (7)

Figure 5 presents the relations of implication between the complex poles and the innermost a-pole. Figure 6 presents the relations of implication between the complex poles and the innermost b-pole.

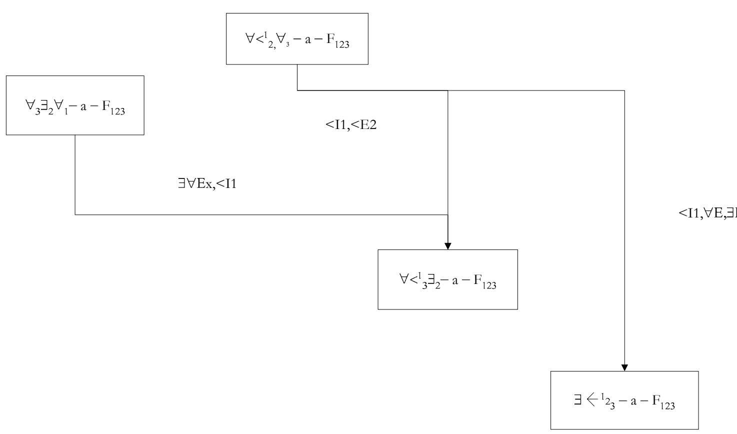


Figure 5: Relations of implication between symbolising poles

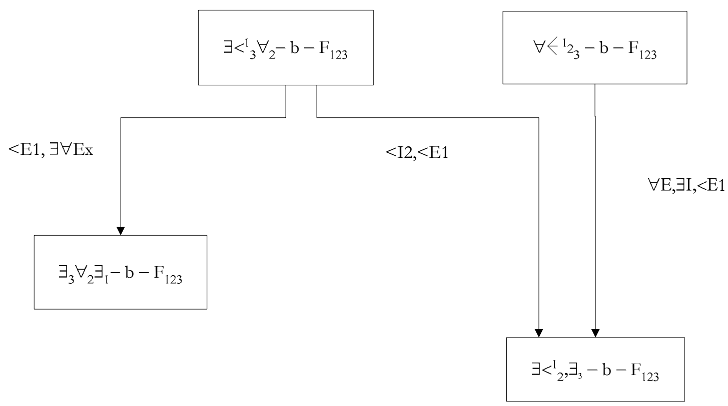


Figure 6: Relations of implication between symbolising poles

According to the definitions of contradictory and contrary poles, contrary poles can be identified by referring to relations of implication between symbolising poles (see Table 9). The first four pairs of poles in the table are contradictory and thus are not only contrary but also subcontrary.

$$
\begin{array}{lcl}
\forall_3\exists_2\forall_1 - a - F_{123} & \lhd\!-\!\rhd & \exists_3\forall_2\exists_1 - b - F_{123} \\
\forall <^{1}_{3}\exists_2 - a - F_{123} & \lhd\!-\!\rhd & \exists <^{1}_{3}\forall_2 - b - F_{123} \\
\forall <^{1}_{2}, \forall_3 - a - F_{123} & \lhd\!-\!\rhd & \exists <^{1}_{2}, \exists_3 - b - F_{123} \\
\exists \lessdot_{\substack{1\\2\\3}} - a - F_{123} & \lhd\!-\!\rhd & \forall \lessdot_{\substack{1\\2\\3}} - b - F_{123} \\
\forall_3\exists_2\forall_1 - a - F_{123} & \lhd\!-\!\rhd & \exists <^{1}_{3}\forall_2 - b - F_{123} \\
\forall <^{1}_{2}, \forall_3 - a - F_{123} & \lhd\!-\!\rhd & \exists <^{1}_{3}\forall_2 - b - F_{123} \\
\forall <^{1}_{2}, \forall_3 - a - F_{123} & \lhd\!-\!\rhd & \forall \lessdot_{\substack{1\\2\\3}} - b - F_{123}.
\end{array}
$$

Table 9: Contrary pairs of poles

Therefore, lines 2, 4, 7-9, 11, 12, and 14 contain contrary poles. In accordance with the rules for generating canonical pole-groups, the pole-groups corresponding to these lines are deleted. Consequently, the following canonical pole-groups remain:

1. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \forall <^{1}_{2}, \forall_3 - a - F_{123},\quad \exists \lessdot_{\substack{1\\2\\3}} - a - F_{123}\},$

3. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \exists <^{1}_{2}, \exists_3 - b - F_{123},\quad \exists <^{1}_{2 \atop 3} - a - F_{123}\},$

5. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \forall <^{1}_{2}, \forall_3 - a - F_{123},\quad \exists <^{1}_{2 \atop 3} - a - F_{123}\},$

10. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \exists <^{1}_{2}, \exists_3 - b - F_{123},\quad \exists <^{1}_{2 \atop 3} - a - F_{123}\},$

15. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \exists <^{1}_{3}\forall_2 - b - F_{123},\quad \exists <^{1}_{2}, \exists_3 - b - F_{123},\quad \exists <^{1}_{2 \atop 3} - a - F_{123}\},$

Table 10: Canonical a-pole-groups of formula (7)

6. $b - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \exists <^{1}_{2}, \exists_3 - b - F_{123},\quad \forall <^{1}_{2 \atop 3} - b - F_{123}\},$

13. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \exists <^{1}_{2}, \exists_3 - b - F_{123},\quad \forall <^{1}_{2 \atop 3} - b - F_{123}\},$

16. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \exists <^{1}_{3}\forall_2 - b - F_{123},\quad \exists <^{1}_{2}, \exists_3 - b - F_{123},\quad \forall <^{1}_{2 \atop 3} - b - F_{123}\}.$

Table 11: Canonical b-pole-groups of formula (7)

The rules for generating the ab-symbol presuppose the identification of subcontrary poles (see Table 12).

$\forall_3\exists_2\forall_1 - a - F_{123}$	$\diamond - \diamond$	$\exists_3\forall_2\exists_1 - b - F_{123}$
$\forall <^{1}_{3}\exists_2 - a - F_{123}$	$\diamond - \diamond$	$\exists <^{1}_{3}\forall_2 - b - F_{123}$
$\forall <^{1}_{2}, \forall_3 - a - F_{123}$	$\diamond - \diamond$	$\exists <^{1}_{2}, \exists_3 - b - F_{123}$
$\exists <^{1}_{2 \atop 3} - a - F_{123}$	$\diamond - \diamond$	$\forall <^{1}_{2 \atop 3} - b - F_{123}$
$\forall <^{1}_{3}\exists_2 - a - F_{123}$	$\diamond - \diamond$	$\exists_3\forall_2\exists_1 - b - F_{123}$
$\forall <^{1}_{3}\exists_2 - a - F_{123}$	$\diamond - \diamond$	$\exists <^{1}_{2}, \exists_3 - b - F_{123}$
$\exists <^{1}_{2 \atop 3} - a - F_{123}$	$\diamond - \diamond$	$\exists <^{1}_{2}, \exists_3 - b - F_{123}$

Table 12: Subcontrary pairs of poles

The merging process adds new pole-groups such that the length of each successively added pole-group is reduced by 1, cf. tables 13 and 14. The application of IR1 and IR2 presupposes the identification of the relations of implication between the poles (see Figure 5 and Figure 6).

1. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\ \forall <^{1}_{2}, \forall_3 - a - F_{123}, \exists <^{1}_{2 \atop 3} - a - F_{123}\},$

2. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\ \exists <^{1}_{2}, \exists_3 - b - F_{123}, \exists <^{1}_{2 \atop 3} - a - F_{123}\},$

3. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\ \forall <^{1}_{2}, \forall_3 - a - F_{123}, \exists <^{1}_{2 \atop 3} - a - F_{123}\},$

4. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\ \exists <^{1}_{2}, \exists_3 - b - F_{123}, \exists <^{1}_{2 \atop 3} - a - F_{123}\},$

5. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \exists <^{1}_{3}\forall_2 - b - F_{123},\ \exists <^{1}_{2}, \exists_3 - b - F_{123}, \exists <^{1}_{2 \atop 3} - a - F_{123}\}.$

6. MR: 1/2 $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\ \ \forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$

7. MR.: 1/3 $a - \{\forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \forall < {}^{1}_{2}, \forall_3 - a - F_{123}, \exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$

8. MR: 2/4 $a - \{\forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists < {}^{1}_{2}, \exists_3 - b - F_{123}, \exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$

9. MR: 3/4 $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$

10. MR: 4/5 $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \exists < {}^{1}_{2}, \exists_3 - b - F_{123}, \exists <^{1}_{\substack{2\\3}} - a - F_{123}\}.$

11. MR: 6/9 $a - \{\forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$

12. MR 8/10 $a - \{\exists < {}^{1}_{2}, \exists_3 - b - F_{123}, \exists <^{1}_{\substack{2\\3}} - a - F_{123}\}.$

13. MR: 11/12 $a - \{\exists <^{1}_{\substack{2\\3}} - a - F_{123}\}.$

Table 13: Merging of a-pole-groups

1. $b - \{\forall_3\exists_2\forall_1 - a - F_{123},\ \ \forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists < {}^{1}_{2}, \exists_3 - b - F_{123}, \forall <^{1}_{\substack{2\\3}} - b - F_{123}\},$

2. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists < {}^{1}_{2}, \exists_3 - b - F_{123}, \forall <^{1}_{\substack{2\\3}} - b - F_{123}\},$

3. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \exists < {}^{1}_{3}\forall_2 - b - F_{123},\ \exists < {}^{1}_{2}, \exists_3 - b - F_{123}, \forall <^{1}_{\substack{2\\3}} - b - F_{123}\}.$

4. MR: 1/2 $b - \{\forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists < {}^{1}_{2}, \exists_3 - b - F_{123}, \forall <^{1}_{\substack{2\\3}} - b - F_{123}\},$

5. MR: 2/3 $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \exists < {}^{1}_{2}, \exists_3 - b - F_{123}, \forall <^{1}_{\substack{2\\3}} - b - F_{123}\}.$

6. MR.: 4/5 $b - \{\exists < {}^{1}_{2}, \exists_3 - b - F_{123}, \forall <^{1}_{\substack{2\\3}} - b - F_{123}\}.$

7. IR1: 6 $b - \{\forall <^{1}_{\substack{2\\3}} - b - F_{123}\}.$

Table 14: Merging of b-pole-groups

IR2 deletes all pole-groups except the prime pole-groups $a - \{\exists <^{1}_{\substack{2\\3}} - a - F_{123}\}$ and $b - \{\forall <^{1}_{\substack{2\\3}} - b - F_{123}\}$. Therefore, the ab-symbol of the initial formula is as follows:

$$a - \{\exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$$
$$b - \{\forall <^{1}_{\substack{2\\3}} - b - F_{123}\}.$$

Paraphrase:
An instance of formula (7)

– is true iff some object, the same in the 1st position, the 2nd position and the 3rd position, makes the 3-adic propositional function F true, and

– is false iff all objects, the same in the 1st position, the 2nd position and the 3rd position, make the 3-adic propositional function F false.

Abbreviations

CL: Wittgenstein, L. 1997. *Cambridge Letters*, Oxford: Blackwell.

MN: Wittgenstein, L. 1979. 'Notes dictated to G.E. Moore in Norway', in *Notebooks 1914 - 1916*, Oxford: Blackwell, pp. 108–19.

NB: Wittgenstein, L. 1979. 'Notebooks', in *Notebooks 1914 - 1916*, Oxford: Blackwell 1979, pp. 1–92.

NL: Wittgenstein, L. 1979. 'Notes on logic', in *Notebooks 1914 - 1916*, Oxford: Blackwell, pp. 93–107.

PR: Wittgenstein, L. 1975. *Philosophical Remarks*, Chicago Press: Chicago.

RFM: Wittgenstein, L. 1967. *Remarks on the Foundations of Mathematics*, M.I.T. Press: Massachusetts.

TLP: Wittgenstein, L. 1994. *Tractatus Logico-philosophicus*, Routledge: London, UK.

VW: Wittgenstein, L. and Waismann, F. 2003 *The Voices of Wittgenstein*, Routledge: London, UK.

WWC: Wittgenstein, L. 1979. *Wittgenstein and the Vienna Circle*, Basil Blackwell: Oxford.

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