Paradoxes and Diagonalization

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Abstract: In this paper Richard's Paradox and the Proof of Cantor's Theorem are compared. It is argued that there is no conclusive reason to treat them differently such as to call the one a Paradox and the other a Proof.

Introduction

The method of diagonalization is at the heart of modern set theory and metamathematics. However, there always have been mathematicians and philosophers opposing to this method and the proofs resting on them. To mention only some: Kronecker, Brower, Poincaré, Weyl, Wittgenstein all did not trust the method of diagonalization. And in our times there are still mathematicians (e.g. Vopenka (1979), Peregrin (1997), Zenkin (2000,2002), Cattabridga (2003), cf. also the often cited paper Hodges (1998) that blames attempts to criticize the diagonal method) as well as philosophers objecting to it (e.g. Gumanski (1986) and the revival of Wittgenstein's point of view in the thorough analysis of Riedecker (2006)). Despite of this fact, great use is made of the diagonal method. This is an understandable fact: The method is simple and gave rise to new branches of mathematics with established proof methods and theorems, it opened up new realms of numbers to discover and seems to make possible to answer philosophical questions with exact mathematical methods. Thus, the method seems to be successful and beyond question. Yet, the so called "success" is not independent of the method's use. Contrary to other branches of mathematics, there is no practical application of the theorems despite of the fact that one stopped searching for solutions of certain mathematical or logical problems (cf. Feferman (1998), p. 30 and chapter 12). The acceptance of some results of the method's use is not a sufficient reason for its rationality. In philosophy we are allowed to articulate doubts and pose questions that are suppressed by mathematical education. Thus, the question of the reliability of the diagonal method should not be ignored.

My aim is to give an argument as simple as possible in order to make understandable why one may doubt the use of the diagonal method. For this sake, I will confine the discussion to Cantor's proof of the nonenumerability of the sets of all sets of natural numbers and Richard's Paradox. In the first part I argue that there is no conclusive argument to treat them unlike. In the second part I go on to present alternative views on diagonalization, only one of them justifying Cantor's Theorem. My aim is not to refute Cantor's Theorem, but to demonstrate that there is a serious challenge for the mathematical tradition to argue for the reliability of the diagonal method and the theorems based on it.

1. Cantor's Theorem and Richard's Paradox

First of all let's recall Cantor's Theorem and Richard's Paradox. Hereby I will try to keep the argumentation as simple as possible.

Put simply, Cantor's Theorem is based on the following three assumptions (*n* is used as a variable, *m* as a constant):

Assumption 1: Enumeration of sets of natural numbers: S_1, S_2, S_3, …

Assumption 2: Definition of the diagonal set $\mathcal{D}$: $n \in \mathcal{D}$ iff. $n \notin S_n$

Assumption 3: $\mathcal{D} = S_m$

Conclusion: $m \in S_m$ iff. $m \notin S_m$

This is a contradiction. Thus, one of the assumptions must be false. Assumption 1 cannot be false, because we can define a method of enumerating sets of natural numbers. Assumption 2 is just a definition and it seems pretty clear how to apply it. Whereas, according to Cantor, Assumption 3 is just what is in question: Is it possible to enumerate ALL sets of natural numbers? Apparently not: Assuming that the diagonal set is part of the enumeration, leads to the mentioned contradiction. Thus, Assumption 3 is false: $\mathcal{D}$ is not part of S_1, S_2, S_3, …- the set of all sets of natural numbers is not enumerable. Anyone, who is familiar with the diagonal method will hardly see any reasonable doubt according to such an admirable proof!

Richard‘s paradox runs as follows:

Assumption 1: Enumeration of definitions of sets of natural numbers: D_1, D_2, D_3, …

Assumption 2: Definition of the set of Richard numbers $S_{\mathcal{R}}$: $n \in S_{\mathcal{R}}$ iff. $n \notin$ of the set defined by D_n

Assumption 3: Definition of $S_{\mathcal{R}} = D_m$

Conclusion: $m \in$ of the set defined by D_m iff. $m \notin$ of the set defined by D_m.

Again, Assumption 1 is unproblematic, because there is a method of enumerating the definitions, e.g. by alphabetic ordering.

The argument is apparently analogous to Cantor's proof. However, it is identified as a paradox. It is not understood as a proof, but as a fallacy based on a mistaken pseudo-definition invoked by some misuse of language. Thus, in case of Richard's Paradox, Assumption 2 – the definition – is refuted.

Contrary to Cantor's Proof, one does not have the choice to reduce Assumption 3 to absurdity without rejecting Assumption 2, because negating Assumption 3 means to negate Assumption 2: Rejecting that the definition of the set of Richard numbers is part of the enumeration of all definitions of sets of numbers, means to reject that the set of Richard numbers is well defined.[1] Thus, in Richard's paradox Assumption 2 must be rejected, whereas in Cantor's Proof only Assumption 3 is reduced to absurdity hereby assuming Assumption 2, i.e. that the definition of the diagonal set is a proper definition. Thus, besides the apparent analogy, there must be a crucial difference of Cantor's Proof and Richard's Paradox that explains why the definition of a diagonal set is trustworthy in one case and not trustworthy in the other.

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[1] Riedecker (2006), p. 108 following Peano (1906), Jackson (1971) and Simmons (1994) argues that also the following implication holds: (B) If the definition of the set of Richard integers is not part of the enumerated definitions, it is a proper definition. This implication is not presumed in this paper. As Peano (1906) pointed out, from (B) another contradiction follows, namely: If the definition of the set of Richard numbers is not a proper definition then it is a proper definition, in Peano's words "If N [the set of Richard numbers] does not exist, then it does exist" (Peano (1906), p. 218). (B) and thus the contradiction derived from (B) is still based on the definition of the set of Richard numbers. Thus, one might still reject (B) by rejecting the definition, e.g. for blaming it not to distinguish sufficiently between meta- and object-language or being impredicative. Thus, it is not correct to say that rejecting that the set of Richard numbers is well defined does not solve Richard's Paradox. Instead one should argue that the solution even solves the Paradox imposed by Peano. One has to keep in mind that rejecting that the set of Richard numbers is well defined does not only mean to reject Assumption 3, but also Assumption 2, i.e. the definition itself and herewith every application of that definition. Furthermore, one cannot say, that if the definition is not part of an enumeration of proper definitions, then it is itself a proper, predicate definition, because the criteria of property of a definition must not depend on being a part of an enumeration of proper definitions.

There is, indeed, a difference: In Richard's Paradox we do not enumerate sets of natural numbers, but *definitions* of sets of natural numbers and we refer to these definitions in the definition of the set of Richard numbers. Now, hasty, the conclusion is drawn all over: This is a confusion of meta- and object-language responsible for a fallacy. Thus, one has what one wants: A position that allows to trust Cantor's Proof and to refute Richard's Paradox.

This strategy to harmonize Cantor's Proof and Richard's Paradox one finds in nearly every standard textbook. Usually, Cantor's Proof has its prominent place in the text, whereas Richard's Paradox is described in the exercise-part. To mention only two examples: In Boolos et al. (2003), p. 21 the description of Richard's Paradox is followed by the innocent question "What is wrong with the Paradox?" given to the student as an exercise. Yet, to this question one gets a rather surprising answer in the solutions of the exercises (Boolos et al. (2003), p. 342):

> "This is a philosophical rather than a mathematical question, and as such does not have a universally agreed-on answer, …"

This is rather unsatisfactory in the light of the close similarity of Cantor's Proof and Richard's Paradox that is pointed out by Boolos et al. themselves. One also is let alone with the question why Richard's Paradox gives rise to philosophical questions whereas Cantor's Proof does not.

However, they go on to give the canonical "explanation" of Richard's Paradox (Boolos et al. (2003), p. 342):

> "… though there is a consensus that defining a set in terms of the notion of definability itself is somehow to blame for the paradox."

Similarly, Delong (1971), p. 256 says:

> „The fallacy consists in the confusion of language and meta-language.“

This blaming Richard’s Paradox for a confusion of language and meta-language seems to be sufficient to stop all discussions. What is worse than not distinguishing sharply between meta- and object-language?

However, on a second look the given explanations are rather question begging.

Delong goes on to explain why the confusion of meta- and object-language is problematic (Delong (1971), p. 256):

> „The property of being Richardian is a property depending on the language and not a property of the natural numbers (whose properties don‘t depend on a particular language). To see this, consider the program of the Richard argument actually carried out in two different languages, say, English and French. Evidently the property of being Richardian-in-French will not be equivalent to being Richardian-in-English because the orderings […] would not be identical.”

In fact, being a Richardian is dependent on the ordering of the definitions and this, again, is dependent on the language of the definitions. Yet, the diagonal sequence in Cantor‘s Proof is also dependent on some arbitrary ordering of the sets of natural numbers and in this sense “not a property of the natural numbers”. Thus, if one blames Richard‘s paradox for confusing properties of natural numbers with properties depending on arbitrary orderings one has to blame Cantor’s Proof for the same reason. Consequently, it remains unclear, what the mistake is that Richard's paradox is based upon but not Cantor's Proof.[2]

The problem of identifying a crucial difference between Richard's Paradox and Cantor's Proof gets even more striking if we consider the correspondence of the definitions of sets of natural numbers and the sets of natural numbers, emphasized by Boolos et al. (2003), p. 21 themselves in their description of Richard's Paradox:

> „[…] we are left with an enumeration of all definitions in English of sets of positive integers, or, replacing each definition by the set it defines, an enumeration of all sets of positive integers that have definitions in English."

Thus we have a function of definitions to sets. Boolos et al. (2003), p.21 go on to explain, that we can "strike out all redundancies of identical sets with different definitions" and thus "obtain an irredundant enumeration of all sets of positive integers that have definitions in English." If one further assumes that Cantor's Proof is not dependent on the idiosyncratic assumption of indefinable sets of natural numbers, we can presume a one-one correspondence of the definitions and the sets in principle. From this, it is not any more intelligible why the definition of the set of Richard numbers is mistaken but the definition of the set

[2] One who does clearly state the question as to the difference of „good" and „bad" diagonal arguments is Simmons (1993), chapter 2 and he applies his distinction to Cantor's Proof and Richard's Paradox: He argues that the diagonal argument in Richard's Paradox is bad, because the enumeration of definitions of sets of natural numbers is not well determined. It is not well determined because the set of Richard numbers is part of the enumeration iff it is not part of the enumeration. Yet, this does also hold for the diagonal object in Cantor's Proof: The Proof shows that if it is part of the enumeration a contradiction follows. If it is not part of the enumeration, the diagonal definition identifies a well defined set of natural numbers and there is no reason not to include it as part of an enumeration of sets of natural numbers. Furthermore, Simmons (1993) does not offer an explanation why, given this situation, one should conclude that the enumeration is not well determined instead of concluding that the definition of the diagonal object is faulty.

$\mathcal{D}$ in Cantor's Proof is unproblematic – the assumptions of both proofs are isomorphic.

Of course, there is a difference: Once we refer to definitions and once to sets – but it is presumed that there is one-one correspondence between them and if one blames Richard's Paradox for a confusion of levels of language, it is not intelligible why one should not equally blame Cantor's Proof for a confusion of classes of different levels. This does not mean to say that this is the right way to deal with both arguments. It only means that there is no reason to treat them differently. Thus, *if* one wants to avoid the contradiction by imposing a hierarchy of languages, one likewise is forced to impose a hierarchy of classes and thus also avoid the contradiction in case of Cantor's Proof.

Thus, in fact, we have arguments of the same structure in both cases with the insignificant difference of the kind of enumerated objects (definitions vs. sets):

> *Assumption 1*: Enumeration of objects.
>
> *Assumption 2*: Definition of a diagonal object.
>
> *Assumption 3*: Defined object is part of the enumeration.
>
> *Conclusion:* The diagonal object is part of the enumeration iff it is not part of the enumeration.

There is no internal reason to treat Cantor's Proof and Richard's Paradox differently.

2. Alternative Views on Diagonalization

Up to now, we have only considered the analogy of Cantor's Proof and Richard's Paradox and did not identify alternative attitudes towards the use of diagonal functions. To this, I will turn now in order to demonstrate that it is not compelling to use the diagonal method like it is used in mathematical proofs. For this sake, it suffices to identify three paradigmatic ways of dealing with diagonal definitions without aiming a complete description of the proposed treatments of the method of diagonalization.

2.1 Platonism

The common use of the diagonal method can be understood as an expression of *Platonism*. A platonistic understanding of diagonal functions presupposes that these are well-defined “perfect genuine total functions” (Boolos et al. (2003), p. 37). It is taken for granted that it is well defined for every natural number whether it is an element of the defined set or not. The set is given – it exists – and it is simply not asked whether it is well defined. Thus, it is presumed that it has a well defined meaning to ask whether *m* is element of the defined set if this set is the *m*’th set in an enumeration. And the derived contradiction simply shows that the well defined set cannot be the *m*’th set, i.e. some set in the enumeration. It does not demonstrate that the defined diagonal function is not well defined.

To grasp Platonism it is helpful to see how it deals with the complement of the defined diagonal function. Given the definition of the complement, no contradiction follows by assuming that the defined set is the *m*’th element of the enumeration of sets of natural numbers. In this case $m \in S_m$ iff $m \in S_m$ holds. According to Platonism either *m* is element of S_m or not, although it is by definition not determined whether $m \in$ of S_m or m $\notin$ S_m and consequently it is impossible to know this.

It is by no means carved out of stone that this attitude towards arithmetical definitions is beyond question. There are established points of view opposing to this platonistic understanding of the definition of diagonal functions.

2.2 The verdict of impredicative functions and type theory

Poincaré's and Russell's "*verdict of impredicative functions*" is an expression of rejecting the definitions of diagonal functions as well-defined. They are not well defined, because they refer to a totality of which the defined object may be a member. This leads to contradictions in case of diagonal objects, because the totality is not given independent of the defined diagonal object itself.

Consider Russell's Paradox: The set of all sets not entailing themselves – does it entail itself? It entails itself iff it does not entail itself. Russell's answer to this is: the set of sets not entailing themselves simply does not exist – it is not well defined, because the totality of "all sets" is not given. In order to define a totality of sets one has to confine the sets to a certain order in a hierarchy of types. The same holds for the set of all sets in Cantor's Paradox. Such definitions of sets seem to be meaningful, but they are not. They seem to be meaningful, because we only have in mind unproblematic applications. Yet, this does not suffice in order to be sure that the definition is applicable in *any* case and thus refers to a well defined *totality*. The case is similar to the barber shaving all men not shaving themselves: Such a barber simply does not exist – the same for the defined sets. The set of Richard numbers is not well defined: The totality of definitions of sets of numbers is not a well defined, because the definition of the set of Richard numbers is a member of this totality if and only if it is not a member of this totality. Thus, the definiens of the definition of the set of Richard numbers does not refer to an unambiguous totality and all that can be concluded is that the definition is a mistaken impredicative definition. Likewise, one can simply deny the definition of the set $\mathcal{D}$ in Cantor's Proof: Instead of presuming that it does exist, but not as part of an enumeration, one can simply deny that it exists, i.e. is well-defined. A criterion is needed in order to identify well defined "sets". Without such a criterion the use of diagonal function is open to doubt. The verdict of impredicative functions defines such a criterion.

In order to avoid paradoxes stemming from impredicative definitions, Russell introduces a hierarchy of the objects one is referring to as it is generally demanded in the case of Richard's Paradox. Russell admitted to be unable to make plausible why his verdict of impredicative definitions that was needed in his theory of classes should not be applied to Cantor's Theorem (cf. Russell (1937), p. 368). Consequently, in the case of Cantor's proof, one might likewise demand that in the definiens of the second assumption the variable *n* cannot take the value *m*. The standard answer to this view is simply that this verdict is too strict, because in this case Cantor's Theorem is not provable anymore (cf. Sainsbury (1995), p.109ff.; Haack (1995), p. 142). This reaction is apparently begging the question.

2.3 Operationalism

There is another view of criticizing diagonal functions hold by Wittgenstein. I call it "*Operationalism*" hereby referring to Wittgenstein's sharp distinction of operations and functions (cf. Wittgenstein (1984), 5.2-5.4, Wittgenstein (1984b), p. 213-219, for Wittgenstein's criticism of the diagonal method cf. Wittgenstein (1984c) II, §1-22 and Riedecker (2006)). Operations are applied iteratively – the result (output) of an operation, in turn, can serve as its basis (input), whereas the argument of a function cannot be its value. The so-called "truth functions" and the so called "number functions" are operations according to Wittgenstein's terminology. Applications of operations are computations, whereas values of functions are not computed, but asserted. Truth and falsehood are only applicable of statements involving functions, whereas it does not make sense to speak of truth and falsehood concerning operations, because there is no corresponding reality to their application. According to Wittgenstein mathematics is based on the concept of operation and not on the concept of functions. At the heart of Wittgenstein's rejection of any platonistic interpretation of mathematics lies his objection to analyse mathematical statements in terms of functions and arguments so that they are capable of being true or false.

Contrary to type theory, Wittgenstein's view does not demand a hierarchy of objects and it does not reject all impredicative definitions. According to Operationalism arithmetical definitions are not definitions of sets of natural numbers but definitions of operations that enable to construct series of natural numbers. Likewise, the definition of a "diagonal function" must be understood as an operation, that, given a list of series of numbers, allows constructing a further diagonal series. Yet, in order to be a well defined operation the "bases" (inputs) of the operation have to be well defined: In case of the “diagonal operation” in Cantor’s Proof or Richard’s Paradox the bases are the n’th elements of the n’th series. Yet, a basis is not defined in case the diagonal operation is applied to the diagonal series itself, because in this case the basis is defined by the result of the diagonal operation and in order to compute this result the basis must be given. Thus the application of the diagonal operation to the diagonal series itself leads to a breakdown of the application. This can be compared to the division by 0 in the system of rational numbers: The operation of dividing natural numbers by natural numbers is applicable in the system of rational numbers but breaks down in case of the division by 0. Likewise, the diagonal operation is applicable within the system of an enumeration of series of natural numbers with the exception of applying it to the diagonal series itself. Instead of concluding that a diagonal series exists, but cannot be enumerated, one should proceed as in the case of division by 0 and infer that assuming, that the diagonal operation applies to itself, violates criteria of a proper definition of an operation.

Both of the described antiplatonistic views reject the use of definitions of diagonal functions and operations, respectively, in indirect proofs. According to both of them one cannot conclude that a non-enumerable set exists, because such a set is not well defined. Instead of inferring that there is a transfinite number of sets of numbers, one should simply say that it does not make sense to speak of a totality of sets of natural numbers.

3. Conclusion

Cantor‘s Theorem is rather an expression of Platonism than a proof of the existence of non-enumerable sets. The challenge for the Platonist is to answer the question: What is the crucial difference of Cantor’s Proof and Richard’s Paradox that justifies regarding the one as a Proof of a Theorem and the other as a Paradox? Unless one cannot meet this challenge and ground Cantor‘s Proof on an objective basis not simply presupposing Platonism as the one and only way of interpreting mathematics, there is no rational basis for the application of the diagonal method.

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