New Logic
Wittgenstein's Alternative

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Part I

Conceptual Foundations

Chapter 1

The main contention

In his early writings, *Notes on Logic* (NL)[1], *Moore Notes* (MN) and *Tractatus logico-philosophicus* (TLP)[2], Wittgenstein compares his conception of logic to "Old Logic". By "Old Logic" he basically means the logic of Frege and Russell. However, as we will see, today's common understanding of first-order logic, mathematical logic in particular, shares the essential features that characterize "Old Logic". In contrast, Wittgenstein's conception of logic stands for a "New Logic". This book systematically elaborates on Wittgenstein's alternative. The first part compares the conceptual foundations of New Logic to the traditional understanding of logic, while the second part works out Wittgenstein's programmatic hints to realize his conception. In doing so, it will be explained why Wittgenstein was justified to regard his conception as a superior paradigm of a prospective logic.

Russell and other contemporaries of Wittgenstein such as Ramsey, Carnap, Schlick or Waismann were attracted by Wittgenstein's new approach. In particular, Wittgenstein was praised for his account of tautologies as being "true by virtue of form". Wittgenstein's new approach was also considered as a contribution to the foundations of mathematics. On the 2nd congress of epistemology of the exact sciences in Königsberg 1930 Wittgenstein's point of view concerning the foundations of mathematics was presented by Waismann as an alternative to logicism, formalism and intuitionism. However, the appreciation of Wittgenstein's programme as a serious alternative to traditional logic and to other conceptions concerning the foundations of mathematics was given up in the following

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[1] Despite the editorial deficiencies of the 1979 edition, I will refer to it because it is used frequently. If necessary, the critical edition Biggs (1996) is taken into account. For a critical edition, cf. also Potter (2009), appendix B.

[2] I refer to the familiar translation by Pears and McGuinness.

decades. This is due to the fact that Wittgenstein's conception of logic and his understanding of mathematics is incompatible with the emergence of mathematical logic. First and foremost this concerns Church's theorem. This theorem was published in 1936 and it states that the property of being a tautology (or theorem) of first-order logic (with or without identity) is undecidable. From Church's theorem it follows that Wittgenstein's outlook of finding a procedure to convert logical formulas to "a notation in which all and only logical equivalents have one and the same representation" is a hopeless endeavor.[3]

However, this judgement does not take into consideration Wittgenstein's critical account of the foundations of modern mathematical logic. According to Wittgenstein's point of view, modern mathematical logic is bedeviled by a confusion from the beginning on. This confusion "pervades the whole of traditional logic" (cf. TLP 4.126). It lies at the heart of "Frege's and Russell's conceptual notation"; this notation still allows for misinterpretations (TLP 3.325). Again and again Wittgenstein refers to this confusion by related distinctions that he accuses "Old Logic" or "the logic of Frege and Russell" not to make: material (or proper) and formal properties / relations (TLP 4.122, PG, p. 476f.), proper and formal concepts (TLP 4.126, WVC p. 224), propositional function and variable (TLP 4.1272, WVC p. 217), function and operation (TLP 5.25, WVC, p. 215), (empirical) totality and system (WVC, p. 213 and 217), descriptions and laws (PR, p. 235), reality and possibility (WVC, p. 214), propositions / statements and rules dealing with signs (WVC, p. 241). These distinctions lie at the heart of what Wittgenstein calls his "main contention", namely "the theory of what can be expressed (gesagt) by prop[osition]s [...] and what can not be expressed by prop[osition]s, but only shown (gezeigt)" (CL, letter 68, p. 124).

According to Wittgenstein's analysis the confusion of what can be expressed by propositions and what can only be shown is induced by the subject-predicate form of ordinary propositions. This form is shared by the notation of function and argument within first-order logic according to its common interpretation. Due to the subject-predicate form of ordinary propositions one does not only state ma-

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[3] cf. Landini (2007), p. 118. Even 13 years before Church's proof, Ramsey was the first doubting the feasibility of Wittgenstein's project in his 1923 critical note to the *Tractatus*, Ramsey (1931), p. 278. He doubted that it was possible to specify a "definite rule" to identify all equivalent formulas. Later he was accompanied, for example, by Black (1964), p. 323, Anscombe (1996), p. 137, Sundholm (1990), p. 60, Floyd (2005), p. 95, Landini (2007), p. 112-118 and, most recently, Potter (2009), p. 181-183 and p. 214. This conviction is even prominent in the editor's comment upon Wittgenstein's writings, cf. the footnote of the editors in *Cambridge Letters* (CL), p. 52.

terial properties of real objects but one also seems to state formal properties of formal structures. Wittgenstein calls propositions of the first sort “real propositions” whereas propositions of the second sort are “pseudo-propositions”. Only real propositions are true or false according to reality. Thus, for example, a propositions such as “$P \vee \neg P$ is a tautology” seems to be capable of being true or false just as it is the case for “Fury is a horse” although, indeed, its truth is not a matter of fact but a matter of representation. According to Wittgenstein, real propositions and pseudo-propositions are not adequately distinguished by distinguishing object- and meta-language. This latter distinction does not distingusih the form of propositions but only the kind of objects that one seems to refer to. Wittgenstein’s calls for a by far more fundamental distinction than that of meta- and object-language. For instance, the distinction he calls for also applies to representing arithmetic properties: as those are formal properties concerning structures of numbers they must not be represented by propositional functions within logic. Instead, those properties are to be represented by proper arithmetic notations representing arithmetic properties by syntactic properties of the number-symbols themselves.

According to Wittgenstein, one cannot *refer* at all to formal structures; one can only represent them by formal structures of the symbolism itself. This, however, is easily overseen as sign-languages such as ordinary language, first-order logic or the figures of the decimal system do not fully express formal structures by structures of their signs. Thus, in order to avoid misunderstandings of propositions involving formal properties and to identify formal properties such as being a tautology, being a certain kind of number or to identify formal relations such as logical implication or identity of arithmetic expressions one must convert the expressions in question to notations that identify those properties and relations by their syntactic properties. In contrast, the way real proposition express properties of objects significantly deviates from the way formal properties are expressed. The main contention of Wittgenstein’s understanding of the language of logic consists in specifying its syntax as the syntax of real propositions. According to him, logic rests on the conception of bipolar propositions that are true or false according to reality (and not according to properties of the formalism itself). Any proper propositional function represents a material property of those objects satisfying the function. Thus, he introduces his ab-notation that assigns two poles, an a- and a b-pole to any proper propositional function. Thus, being a material property that is not decidable by syntactic means is itself a formal property of this symbolism. It is this notation to which any formula of first-order logic must be converted to in order to specify its formal properties properly.

What formally true or false pseudo-propositions intend to say, however, is not expressible by propositions within the language of logic. Instead, it is "shown" by the properties of an adequate representation of the formal structures to which those properties are attributed by pseudo-propositions. Thus all tautologies of first-order logic must be identifiable by a common structural property within a proper logical notation; it must be possible to reduce all tautologies to one representative in the ab-notation or any other isomorphic symbolism. The all important difference between real propositions and pseudo-propositions is that the truth (or falsehood) of the first sort depends on some non-symbolic reality whereas the truth (or falsehood) of the second sort is decidable by translating them to a proper notation. Decidability thus becomes a matter of adequate representation. In case of first-order logic decidability becomes a question of defining an optimized equivalence procedure such that all equivalent formulas are reduced to one representative.

According to this point of view, undecidability theorems rest upon the confusion of real propositions and formally true or false pseudo-propositions. These two kinds of propositions are not distinguished syntactically in Old Logic as pseudo-propositions are just as well as real propositions represented within the language of traditional logic. This is the key mistake of Old Logic according to Wittgenstein's analysis. Church's theorem is just one consequence of a fundamental "category mistake" that "Old Logic" is based on. This, for example, becomes evident by Church's thesis which states that any decidable formal property is represented by a characteristic primitive recursive function, which, in turn, is representable by a propositional function within the language of logic. However, Church's thesis is only one example of a by far more general confusion that "pervades the whole of traditional logic". This confusion is implemented in the syntax of traditional first-order logic as this syntax does not distinguish between material and formal properties. Thus, nothing prevents from representing material as well as formal properties by propositional functions, $\varphi(x)$. Thus, instead of representing formal properties by syntactic properties of the signs themselves, this induces misinterpreting propositional functions within the language of logic by formal properties. Instead of proving undecidability, undecidability proofs first and foremost demonstrate the inadequacy of representing mathematical and metamathematical propositions within the language of logic.

According to Wittgenstein's analysis, modern mathematical logic is based on misunderstanding the conditions of the possibility to represent formal structures. New Logic deviates from modern mathematical logic by neglecting the possibility of representing formal structures adequately by interpreting propositional functions within a formalism of logic. Instead, the adequate representation of formal structures must reduce them to formal structures of a proper notation itself. This is what Wittgenstein means by saying that formal properties cannot be expressed by propositions but only shown by formal properties themselves, which is to be realized by inventing proper notations. Thus, it is impossible to speak properly about formal properties within the language of logic. Instead, any adequate representation of formal properties must be able to reduce them to syntactic properties, i.e. properties of the structure of an adequate notation. To presume the contrary induces fallacies. The first part of this book spells out this critique of modern mathematical logic in all details.

The second part of this book then shows how to realize Wittgenstein's conception of identifying formal properties in the realm of logic. It refutes Church's theorem by defining an equivalence procedure that allows one to identify all and only logical equivalents of pure first-order logic formulas by one and the same symbol. This procedure will be spelled out in two ways: (i) on the basis of nothing but well-known derivation rules of first-order logic and (ii) in terms of specifying all the details of Wittgenstein's ab-notation. (i) reduces Church's theorem to absurdity on the basis of nothing but well-known, ordinary syntax of first-order logic. Thus, (i) does not *depend* on any kind of interpreting the logical symbolism although it is *motivated* by a completely different account of representing formal properties of a logical symbolism. However, (i) answers the question of the decidability of first-order logic to the positive independent on the acceptance or rejection of the traditional interpretation of the logical symbolism that is presumed by undecidability proofs. Thus, one should accept the decidability of first-order logic on the basis of a small part of syntactic presumptions of traditional logic although this conflicts with other parts of traditional logic, namely Church's theorem and its underlying interpretation of the logical formalism. (ii), in turn, refutes the traditional interpretation of first-order logic by establishing a notation for first-order logic that makes misinterpretations impossible. It identifies inadmissible interpretations as the source of faulty metamathematical proofs and thus explains what goes wrong in metamathematical proof methods. Misinterpreting first-order formulas is at the heart of mathematical logic and its theorems. This goes hand in hand with misinterpreting mathematical equations. Wittgenstein's point of view does not allow for going beyond the formalism of logic and the formalism of pure mathematical equations and equivalence procedures within these formalisms. How formulas are to be "interpreted" must be based on nothing but pure formal equivalence procedures within the formalism. Mathematical logic, however, goes beyond this border. On the basis of a broad concept of a function it allows for interpretations going beyond the formalisms and establishing new kinds of proofs and proof methods. Although it is mostly overseen nowadays, it these proofs methods and their underlying technique of interpreting mathematical and logical expressions is questionable and in need of a justification. What could be more appropriate to decide upon the correctness of meta-mathematical proof methods than the unquestioned basis of the formalisms themselves and the equivalence procedures carried out within them?

The book's intention is to refute the traditional interpretation of first-order logic and the reconstruction of mathematics within the language of first-order logic by defining a decision procedure for first-order logic on the basis of nothing but established equivalence rules of first-order logic. In doing this, I do no more than spelling out the consequences following from Wittgenstein's "main contention".

1.1 Truth as a primitive notion

According to Frege, science aims for truth. Logic is the proper language to speak about truth. Frege intended to invent a logical notation by striping of all elements of ordinary language that are inessential for speaking about truth and falsehood. In contrast to questions, commands or wishes, propositions are either true or false. Frege treats propositions as names that refer to truth values: the True or the False. According to his philosophy of language, propositions also have sense (meaning). However, logic abstains from the sense of propositions. The essential motivation for introducing the formal language of logic to science is to invent a criterion of proof. If we write down a proof within the logical formalism we can apply the logical calculus to judge upon the validity of the purported proof. The proof is valid if and only if it preserves truth. The truths of science can be proven by deriving propositions from axioms that are presumed to be true without further proof. Frege, Peano and Russell applied the language of logic to mathematical propositions. They based logic on the extensional concept of a function. A function assigns objects to objects. While Frege and Russell intended to define classes by propositional functions and to define numbers as classes of classes, Peano and modern mathematical logic treat numbers as objects. Thus, in modern mathematical logic numerical functions assign numbers to numbers. Propositional functions about numbers assign truth values to tuples of numbers. Truth functions assign truth values to truth values. Arithmetic propositions are analyzed as truth functions of propositions about numbers.

Within this traditional conception truth is a primitive notion. At best, true propositions can be captured by derivation from true axioms. Within the axiomatic conception of mathematics it becomes hard to specify the analytic character of mathematical propositions. Logicism tried to do so by deriving mathematical propositions from purely logical axioms. However, no syntactic criterion is available to identify the truth of axioms. Furthermore, specifying logical truth by theoremhood presumes a correct calculus at least and a correct and complete calculus at best. This, in turn, presumes that the calculus "preserves truth" and "captures truth". Thus, any calculus is assessed by its capability to derive true propositions and not vice versa. Finally, in a common calculus based on first-order logic, formulas that are not theorems cannot be identified as not being theorems because the calculus only allows one to identify theorems in a finite number of steps. Besides defining well formed formulas, syntax is reduced to specifying derivation rules. The idea of converting logical or mathematical formulas to ideal symbols that provide syntactic criteria to identify logical or mathematical properties is absent.

Despite of specifying logical truth as theoremhood, all one can do within traditional logic is to characterize logical truth semantically by "truth relative to all interpretations". However, any semantic definition of logical truth does not provide a criterion within first-order logic as one has to refer to an infinite domain as soon as one exceeds propositional logic according to traditional semantics. Furthermore, the semantic characterization of logical truth does not apply to propositions but to formulas that are interpreted. Thus, strictly speaking, one does not call propositions such as "It is raining or it is not raining" logically true. Instead, it is the formula $P \vee \neg P$ that is logically true according to the semantic definition as it is this formula that is true relative to all interpretations of P. To interpret P by the truth value of "It is raining" is only one possible interpretation. Finally, the fact that an infinite number of logically equivalent formulas exists within first-order logic makes evident the absence of a syntactic criterion of logical truth. Propositions instantiating $P \vee \neg P$ and propositions instantiating $\neg(P \wedge \neg P)$ are said to have different logical forms although both formulas are logically true. Thus, to say that propositions are "true by virtue of form" if they instantiate a logically true formula does not identify a common syntactic property that would explicate logical truth. Rather, by defining logical or formal truth as truth according to all interpretations, logically true formulas are identified by referring to truth values of their instantiations. Not formal properties of a certain notation but the truth and falsehood of propositions is the primitive concept. Thus, Russell concluded rightly that traditional logic does not provide a satisfactory explanation "of what is meant by saying that a proposition is 'true by virtue of its form'" (Russell (1937), p. xvi).

According to Wittgenstein, logic does not presume the truth or falsehood of any kind of proposition: neither of axioms nor of interpretations of logical formulas. Treating truth as a primitive notion is just a consequence of the confusion of material and formal properties that prevails Old Logic. It goes hand in hand with "Russell's chief mistake", namely "that he time and again tries to reduce possibility to reality" (WVC, p. 214). Logic is not concerned with the truth or falsehood of propositions but only with their *possibilities* of being true and false. These possibilities are determined by the form of propositions. The task of logic is to specify these possibilities on purely formal grounds; it provides syntactic criteria of truth-possibilities. As we will see, this presumes a certain interpretation of the form of the logical symbolism. Yet, it does not presume any kind of interpretation in terms of assigning special meanings or certain extensions (truth values, objects or sets of tuples of objects) to the logical symbols; New Logic is independent of any kind of traditional semantics.

Wittgenstein accuses traditional logic of not being able to distinguish by virtue of form (i) between propositions with certain truth-possibilities and propositions not involving truth-possibilities and (ii) between different truth-possibilities of propositions of the first sort. Propositions with certain truth-possibilities are either meaningful ("sinnvoll") or they lack sense ("sinnlos"). Meaningful propositions are materially true as they allow for both possibilities: being true and being false. They are true by virtue of (symbolized) facts. Propositions lacking sense are either tautologies or contradictions (TLP 4.461); in the first case they are logically true, in the second case they are logically false. Tautologies and contradictions are still representable within the logical symbolism as they involve truth-possibilities (TLP 4.4611). They are the two extreme cases of truth-possibilities (TLP 4.46). By virtue of their form tautologies lack the possibility of being false, whereas contradictions lack the possibility of being true. Logic must specify these forms as well as any other forms of truth-possibilities; it defines systematically truth-possibilities. Logically true or false propositions are true or false according to the logical formalism; they are a special sort of formally true or false propositions.

In contrast, propositions not based on truth-possibilities are meaningless or "nonsensical" ("unsinnig"). Those propositions are not representable within the logical symbolism. Calling them "nonsensical" (meaningless) is to be taken literally: Unlike meaningful propositions, they do not "stand in any representational relation to reality" nor do the "conditions of agreement or disagreement with the world – the representational relations – cancel one another out" (TLP 4.462) as it is the case for tautologies or contradictions. Nonsensical proposition can, in turn, be classified into those that are still formally true or false and those that are not true or false in any sense. Nonsensical propositions of the first sort are not logically true or false as they are not representable within the logical formalism. Thus, they are not true or false by virtue of logical form but, for example, by virtue of the form of some proper arithmetic notation. Thus, for instance, equations are formally true or false though they are not logically true or false. As we will see, there are also other notations than logical or mathematical notations that identify formal properties that might be expressed by nonsensical propositions of this sort. Thus, for example, diagrams, maps, paintings or written notes might just as well identify formal properties. The truth of nonsensical propositions of the first sort consists in a correct identification of some formal property, e.g. as the result of some calculation expressed by a manipulation of signs. It does not consist in some correspondence to facts expressed by a statement about material properties of denoted objects. The falsity of this sort of nonsensical propositions means some misidentification of a formal property, e.g. as the result of some mistaken calculation; it does neither mean some conflict with non-symbolic reality nor some fundamental misconception of the properties or rules of a notation, e.g. the rules of equivalence transformations of equations. The "mistaken calculation" rather consist in a mistaken application of known and accepted rules than in a misconception of those rules.

Nonsensical propositions of the second sort, which are not true or false at all, use meaningful symbols in a way that does not correspond to their proper syntactic use in materially or formally true or false propositions. Wittgenstein calls propositions of this kind "antisyntactic"[4]. He does not refer to syntax in terms of the rules of ordinary grammar but to syntax in terms of rules constituting meaningful or formally true or false propositions. These "antisyntactic" propositions are problematic as they seem to be legitimate propositions according to ordinary grammar. They are grammatically well-formed subject/predicate propositions that can be said to be true or false without violating the syntax of ordinary language. However, they are illegitime according to a more thorough analysis that is not based on the surface structure of ordinary language but on the use of symbols and how they contribute to determine truth or falsehood. This analysis rather determines what it means to say in one or another profound way that some proposition is true or false instead of simply accepting truth or falsehood of grammatically well-formed propositions as primitive. It diagnoses ordinary language as a deficient notation that neither expresses differences of the syntactic employment of symbols by differences of their surface structure nor expresses formal equivalences by identity of

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[4] cf. the German edition of WVC, p. 220. The English translation ignores the sentence "Unsinn sein heißt antisyntaktisch sein" ("Nonsense means to be antisyntactic").

surface structure. Thus, ordinary language must be replaced with proper notations that express the proper syntactic use of the symbols by properties and relations of their signs. The possibility of syntactic misunderstanding and thus of antisyntactic propositions is caused by the insufficient surface structure of ordinary language. Deficient distinctions cause syntactic confusions.

Philosophy is clarification of the syntax (WVC, p. 220). For this sake, it must analyze the syntax of meaningful propositions as well as the syntax of formally true or false propositions. However, first and foremost this serves to diagnose and dissolve the second sort of nonsensical propositions that are neither true nor false in any substantial meaning. For the whole project of Wittgenstein's analysis of language it is cardinal to come familiar with this possibility of "antisyntactic" propositions. Wittgenstein's criticism of mathematical logic to confuse material and formal properties is nothing but an application of his analysis of propositions that accuses mathematical logic to confuse real propositions and propositions that are formally true or false. From this, in turn, nonsensical propositions of the problematic sort arise. Analyzing mathematical equations as real propositions that are representable within the language of logic is just one example hereof. Church's theorem, which is based on analyzing formal properties such as being a tautology in terms of primitive recursive functions (Church's thesis), is another example. I will elaborate this in detail in the following chapters. For now, it may suffice to illustrate the existence of nonsensical propositions of the second sort and the kind of their analysis by a simple example. In TLP 4.1272 Wittgenstein mentions "2 + 2 equals 4 at 3 o'clock" as an example of a nonsensical propositions. According to its grammatical form this is a proposition that might be called "true" or "false". However, it would be off the point to judge upon the truth or falsehood of such a proposition in any substantial meaning. One can only state that it is meaningless to attribute time specification to equations. The expression "2 + 2 = 4" is a proper symbol that has its specific use within a calculus of arithmetic. Likewise, the symbol "at 3 o'clock" is meaningful within real propositions stating events or processes. Yet, to combine those two expressions is in conflict with their ordinary, proper use and does not constitute any proposition that might in any profound meaning be true or false. One confuses the syntax of different contexts in which those symbols are used properly: The syntax of an arithmetic notation identifying formally true equations is confused with the syntax of meaningful propositions about events or processes. This possibility of syntactic confusions is induced by ordinary grammar that does not call for more than well-formed subject/predicate propositions. According to the surface structure of ordinary grammar the proposition in question has just the same form as "John and Jill meet Peter at 3 o'clock".

In order to avoid a confused employment of symbols one must specify their proper syntactic use by specifying proper notations that make it impossible to articulate nonsense of this kind. The question is not whether there are “antisyntactic” propositions. Rather, the question is which propositions one identifies as propositions of this sort. According to Wittgenstein, traditional philosophy as well as modern foundations of science is full of propositions based on syntactic confusions. Mathematical logic does not mark an exception.

I will call the first sort of nonsensical propositions “non-sensical”; there is nothing bad or confused in this kind of propositions. It is an incorrect analysis of propositions of this kind as well as of logically true or false propositions that induce nonsensical proposition of the second sort. I call this second sort of nonsensical propositions “nonsensical” in the strict sense or, likewise, “antisyntactic” propositions. It is only this sort of nonsensical propositions that indeed are mistaken or confused.

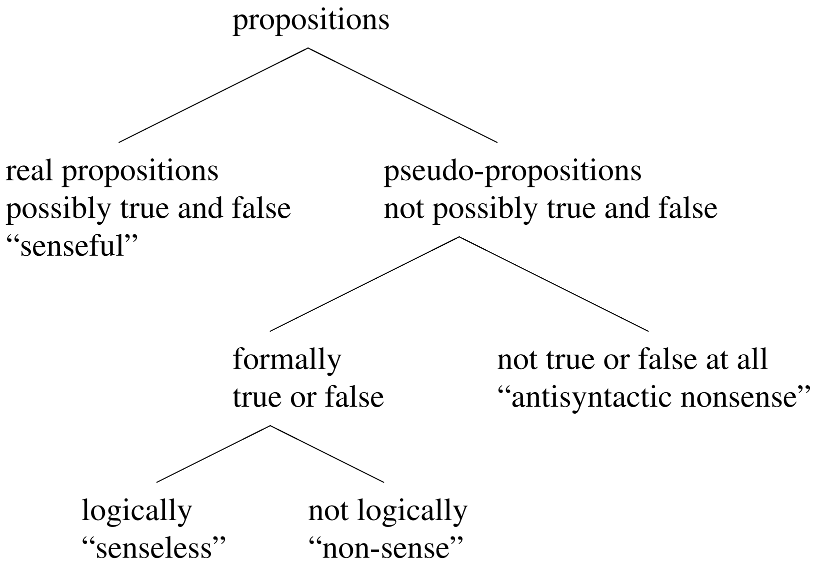

Figure 1.1: Classification of Propositions

1.2 Function vs. Operation

Before we go on to characterize Wittgenstein’s general conception of syntactic analysis it will be helpful to illustrate his point of view in case of mathematics.

This explains his fundamental rejection of mathematical logic and why one is motivated to look for decision procedures despite of undecidability theorems from a Wittgensteinian point of view. One way to account for Wittgenstein's rejection of Old Logic is tracing it back to his distinction of function and operation. This objects to base mathematics on the broad concept of extensional functions. Instead, Wittgenstein aims for basing mathematics to his concept of an operation. For now, the all important point to come to see is that Wittgenstein's rejection of the extensional concept of a function in mathematics is based on the fact that this concept exceeds the limits of a purely formal account of mathematics. According to his point of view this does not only induce a misconception of mathematics but also makes it impossible to give an adequate account of computation and its related concepts such as decidability or effective enumerability.

In sections 1.3 to 1.6 I will characterize Wittgenstein's general conception of syntactic analysis and of interpreting logical formulas before we apply this to first-order logic in chapter 2. Before, I will illustrate some basic differences of mathematical logic and Wittgenstein's syntactic analysis of propositions by the analysis of trivial arithmetic equations. The intention of our discussion in this section is not to say something profound on arithmetic equations. Instead, I only intend to illustrate why the logical formalization of equations fails to do justice to Wittgenstein's standards.

According to the traditional point of view, arithmetic equations such as $2 = 2$, $1 + 1 = 2$ or $2 \times 3 = 6$ are simply true propositions. As the principle of bivalence is satisfied, this suffices for their representability within the language of logic according to the standards of traditional logic. Within propositional logic, $2 = 2$ is a proper instantiation of an atomic propositional variable P. In this case, P is interpreted as the name of the truth value TRUE as $2 = 2$ is true. Within first-order logic, $2 = 2$, or $=(2{,}2)$ in strict logical notation, is analyzed as a relational statement, composed of the dyadic predicate "=" and the name "2".[5] Thus, the statement is of the form $\varphi(x{,}x)$. "=" is an instantiation of a dyadic predicate and interpreted as referring to the class of pairs with identical relata. "2" is a name, referring to an object, namely the number 2. According to this analysis $2 = 2$ cannot be identified as "true by virtue of form" as the form $\varphi(x{,}x)$ of this statement may also be instantiated by false propositions such as $>(2{,}2)$. This syntactic analysis of the proposition $2 = 2$ does not identify a syntactic feature that allows one to identify the truth value of $2 = 2$. This does not change within first-order logic with identity or any axiomatic system of arithmetic such as Robinson

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[5] We only use double quotes to refer to symbols if italic- or math-font is not obvious.

Arithmetic or Peano Arithmetic. Here, “=” is introduced as a logical constant with a fixed interpretation and identity axioms such as $\forall x\, x = x$ are introduced in order to prove true identity statements such as $2 = 2$. It is not syntactic features of the formal representation but axioms and fixed interpretations that identify the truth value of $2 = 2$. This way of a formal representation within axiomatic systems of logic intends to distinguish true and false propositions. Yet, it does not provide means to distinguish formally true and materially true propositions.

Wittgenstein’s syntactic analysis of equations such as $2 = 2$ disagrees with the traditional representation of equations within logic. Contrary to the traditional understanding, Wittgenstein does not treat truth or falsehood of arithmetic equations as a sufficient criterion for their logical representability. Instead, a proper formal representation must distinguish between formally and materially true propositions. The logical formalization of arithmetic equations does not satisfy this standard.

According to Wittgenstein’s analysis arithmetic equations (without variables) are formally, but not logically, true (or false). This is due to his interpretation of number symbols (numerals). He does neither interpret numerals as names that refer to objects nor does he define their meaning by descriptions as Frege and Russell do. Instead, he interprets numerals such as digits of the decimal system as systematic abbreviations of forms, which are construed by operations. Operations are syntactic devices to vary forms by iteration; they determine formal properties. Natural numbers, for example, are definable by the successive application of the operation $+1$ (TLP 6.02-6.03). Likewise, Wittgenstein rejects to define real numbers as infinite sequences of rational numbers (Cauchy sequences) or to identify them simply by descriptions such as “the number that multiplied by itself is identical with 2”. Instead, he defines proper real numbers by operations, which successively generate a series of forms visible within a proper symbolic representation. Thus, e.g., he defines $\sqrt{2}$ by the regular continued fraction $[1, \overline{2}]$, i.e. by the sequence $1 + \frac{1}{2}, 1 + \frac{1}{2+\frac{1}{2}}, 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \ldots$ (cf. MS 107, p.99). This definition is based on equivalence transformation starting with $x^2 = 2$ and resulting in $\mathbf{x} - \mathbf{1} = \frac{1}{2+(\mathbf{x}-\mathbf{1})}$. Thus, $\sqrt{2} - 1$ is representable by the operation $\frac{1}{2+(\mathbf{x}-\mathbf{1})}$. The original description of $\sqrt{2}$ by $x^2 = 2$ is reduced to a purely syntactic definition of a formal series. Obviously, the sequence generated by the operation $\frac{1}{2+(x-1)}$ can be construed by successive variation of a formal property of the signs themselves, whereas the representation of $\sqrt{2}$ within the decimal system does not make visible any law within the notation. The decimal notation does not identify numbers and their relations by syntactic properties. That is why this representation of numbers is deficient. This, of course, does not mean that decimals or common descriptions of numbers cannot be used properly. It only means that this kind of representations are open to misinterpretations such as the conception of “infinite sequences in extension” or “non-denumerable classes of numbers”. Such misunderstandings are ruled out by converting deficient representations to proper notations revealing formal properties and relations. For now, it suffices to come to see that Wittgenstein interprets number symbols as devices to generate a syntactic structure, i.e. a form or series of forms. It is this form that identifies the respective number. Any proper, formal representation reveals this form and thus identifies numbers as well as their relations by purely syntactic properties and relations.

This understanding of numbers cuts of the attempt of a logical representation of arithmetic. It rules out any kind of arithmetic interpretation of the logical formalism, namely (i) to interpret names of the logical symbols by numbers and, in consequence, (ii) to lay down a certain “set of numbers” such as the natural numbers as domain and to quantify over numbers in a logical sense, (iii) to interpret predicates of the logical symbolism by arithmetic properties or relations, and (iv) to interpret logical formulas by arithmetic propositions. Neither arithmetic equations nor propositions about certain properties or relations of numbers, e.g. “Goldbach’s conjecture”, are adequately represented within the logical symbolism according to this conception. Wittgenstein deems the language of logic the language of propositions having the form to state facts. A logical calculus determines internal relations between propositions of this form. Equations are no meaningful propositions stating some fact beyond syntax; they are non-sensical pseudo-propositions (TLP 6.2f.). Equations are neither part of the logical language, nor does the calculus of logic suffice to identify internal relations between numbers. Solutions of equations apply specific equivalence transformations of arithmetic, which are based on arithmetic operations, e.g. addition, multiplication, subtraction or division, and not on logical operations (cf. TLP 6.22f., WVC, p. 218f.). Equations with variables, e.g. $(x + 2)^2 = 16$, can be solved by equivalence transformation, which shows that the variable in such equations is used as an “unknown” that has computable values (cf. WVC, p. 108f.). Variables in arithmetic equations are not used as variables within proper propositional functions, which might be bound by quantifiers resulting in a logical formula that at best follows from some axioms. Arithmetic propositions call for specific arithmetic notations that allow one to identify properties and relations of numbers by their syntactic representation (cf. TLP 6.232). Wittgenstein does not interpret arithmetic signs such as $+$ or $\times$ as arithmetic functions assigning objects to objects but as arithmetic operations determining syntactic features of forms. Thus, for example, 2×3 is interpreted as the instruction to write down the form $1 + 1$ three times, which results in a new form, namely $1 + 1 + 1 + 1 + 1 + 1$. Identity of arithmetic terms, thus, becomes a formal relation identifiable by the proper syntactic representation of the terms in question. It follows that identity of numbers or arithmetic expressions is not represented adequately by a dyadic predicate. In sum, no logical calculus is needed to prove arithmetic propositions. Arithmetic is no system of meaningful propositions, it is computing formal properties of numbers. This means to manipulate the form of symbols; it does not mean to state facts. It is the ordinary form of arithmetic propositions that misleadingly suggests their logical formalization. Yet, this formalization does not do justice to the use of arithmetic propositions within arithmetic calculations.

To be clear: At this stage, I merely want to point out why Wittgenstein rejects logical formalization of arithmetics. For now, it suffices to come to see that it is syntactic analysis of propositions that is at the heart of the differences between "Old" and "New Logic". For Wittgenstein such an analysis must satisfy the demand to identify formal truths by the properties of an adequate notation. Before I apply this to the analysis of the form of propositions representable within logic in chapters **??** to 2, I specify Wittgenstein's general conception of syntactic analysis or "interpretation of the form of a symbolism" (MN, p. 114[4]) as he calls it in the following sections.

1.3 Syntactic analysis

For Wittgenstein, interpreting a symbolism is not reducible to fixing the meaning of signs in terms of their reference. He objects to reducing interpretation to one kind of interpreting symbols, namely to determine their reference. This reductive view of interpretation distinguishes different types of objects that symbols refer to instead of distinguishing different kinds of forms of symbols which correspond to "different modes of signification" (cf. TLP 3.322 and 3.33, CL, letter 16). A symbol is not merely a sign with a certain meaning. Instead, it characterizes both, a form and a content (TLP 3.31). One cannot refer to forms; a form is what the symbol must have in common with whatever it represents. Prior to assigning a special meaning to a sign, the form of the symbol must be determined. The form of a symbol, however, is not merely determined by the sign but only by the sign with its "syntactic employment" (TLP 3.327). The same sign used in a different way is not the same symbol (TLP 3.321-3.323). The task of the interpretation of the form of symbols is to identify the form of symbols. To identify the form of symbols, one must take into consideration how a sign is used in order to contribute to the sense of propositions (TLP 3.326). The proper, adequate form of a symbol is the form constituting its use. In investigating the use of symbols, misinterpretations are possible.

One cannot deny this possibility of misinterpretation either by referring to the surface grammar of ordinary language as a criterion to identify the form of symbols or by referring to the familiar syntax of first-order logic and its common interpretation. This is so because the adequacy of the identification of the form of symbols according to the surface of ordinary or formal languages is just part of what is in question. To mention only one sort of example: That grammatical predicates such as "_ is prime", "_ is identical with _", "_ is a natural number", "_ causes _", "_ is a fact", "_ is the same fact as _", "_ is necessary", "_ is logically true", "_ is provable" are representable by propositional functions within the language of logic must not be presumed. Questions like the adequacy of representing grammatical predicates by logical predicates are in danger to be suppressed by customization or by answering them in one way without considering unfamiliar though reasonable alternatives. The use of the language of logic ever so often is more or less based on a rather naive interpretation in the light of ordinary language. Wittgenstein challenges this practice by assessing the use of formal languages according to his criterion of distinguishing material and formal properties due to identifying the latter by syntactic properties of an adequate notation. It makes a difference in the use of propositions if they must be compared to reality (or some standard beyond syntactic manipulation) to judge upon their truth or not. Thus, taking the superficial form of ordinary propositions as criterion of determining their form does not do justice to their use. In consequence, it does not identify their form properly.

Deficient notations do not identify the form of the symbols by virtue of syntactic criteria in terms of perceivable properties of the signs. This induces misinterpretations of the use of symbols. The result of a proper interpretation of the form of symbols is a proper notation. The task of a proper notation is to avoid misinterpretations of the form of symbols by use of a sign-language that identifies the form of the symbols by its properties (TLP 3.325). Ordinary language is a misleading language open for misinterpretation as it uses similar signs for different kind of symbols and differently structured signs for the same kind of symbols. A proper logical notation should overcome these deficiencies. However, according to Wittgenstein's standards, the "conceptual notation of Frege and Russell", and thus common "sign-languages" of first-order logic in general, still share deficiencies with ordinary language (TLP 3.325). Thus, it fails to exclude all mistaken interpretations. Contrary to the understanding of the common language of first-order logic in terms of an ideal language that identifies the logical form of propositions, Wittgenstein regards the common syntax of expressions of first-order logic still in need of a further analysis. First-order formulas do not identify the logical form of propositions satisfactorily. This does not mean that the language of first-order logic or ordinary language are not “in perfect logical order” (cf. TLP 5.5563). It only means that these languages are open to misinterpretations due to syntactic confusions as the form of the symbols is not identifiable by perceivable, syntactic properties of the signs of these languages. It is only a proper notation that provides syntactic criteria for the form of symbols. However, in so far as a deficient notation is convertible to a proper notation the original possibility of misinterpretations is omitted and as far as it is used according to its proper formal interpretation the deficient notation is “in perfect logical order”. We will elaborate on Wittgenstein’s ab-notation as the notation for first-order formulas that brings to light its proper interpretation.

Different kinds of symbols are to be represented by symbols with different syntactic properties of the signs in a proper notation. Interpreting symbols in terms of assigning special meaning to symbols is subsequent to the interpretation of the form of symbols. According to Wittgenstein’s terminology, “what symbolizes” (cf. MN p. 114[5]) is not the same in case of different kind of symbols.[6] The “symbolizing properties” of different symbols, or, in case of a logical notation, their “logical properties” differ. Wittgenstein distinguishes the interpretation of the form of a symbolism and the interpretation in terms of giving meaning to the signs or “particular scratches” of a symbolism, MN, p. 114[4]:

> The important thing is that the interpretation of the form of the symbolism must be fixed by giving an interpretation to its *logical properties, not* by giving interpretations to particular scratches.

Signs are not simply arbitrary scratches that bear the burden of whatever interpretation. Instead, different kinds of interpretation presume different forms of symbols. Thus, prior to assigning special meanings to the symbols the relevant syntactic features and the way how these features contribute to the meaning of propositions must be determined. This is done by the “interpretation of the form

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[6] Wittgenstein uses the verb “to symbolize” as a technical term in NL and MN. In contrast to its common usage, he uses this verb as an intransitive verb. Thus, he also makes use of the termini “symbolizing property / feature” and “symbolizing fact”. We adopt Wittgenstein’s use of the verb “to symbolize” in this book.

of symbols", or, in other words, by "syntactic analysis". Thus, for example, although the scratches "1", "a", "P" seem alike, their use as a number symbol, a name or a proposition enforce different interpretations of their form as we will see. These difference must become transparent in a proper notation. The same holds, for instance, in case of symbols like "$2 = 2$", and propositions instantiating "$q \vee q$" and "aRa" and their use as an equation, a truth function or a contingent proposition. The signs may not differ essentially according to their (deficient) syntax. Yet, this only shows that determining their proper syntax is a matter of interpretation or further analysis. Traditional logic, however, accepts the syntax of first-order logic at its face value.

To represent different forms of symbols by different syntactic features is only one standard of a proper notation. Another standard is to represent identical forms of symbols by identical symbolizing properties. The distinction between symbolizing and non-symbolizing properties is fundamental to adequately interpreting a proper notation, cf. NL, p. 99[2]:

> In regard to notation, it is important to note that not every feature of a symbol symbolizes.

Not any syntactic feature of a proper notation must be significant for identifying formal properties. Yet, it must be possible to distinguish those syntactic properties that are insignificant and those that serve as identity criteria for formal properties. In case of a logical notation, it must be possible to identify those syntactic properties that serve as identity criteria for determining truth-possibilities. In sum, a proper notation must have the adequate "syntactic manifold"; any and only differences in form must be expressed by differences in symbolizing properties. The resulting expressions must make it possible to read off directly the form of the symbol in question. Different signs with the same meaning and, consequently, any two equivalent propositions, cannot signify in different ways. Although equivalent signs might differ according to their ordinary syntax, their symbolizing features must be identical. That is why "all and only logical equivalents must have one and the same representation" (Landini (2007), p. 118) in a proper notation. A proper logical notation must provide the means to identify truth conditions by its syntactic features. This standard is not satisfied by first-order formulas as an infinite number of different first-order formulas represent equivalent propositions, cf. NL, p. 102[3]:

> If p = *not–not–p* etc.; this shows that the traditional method of symbolism is wrong, since it allows a plurality of symbols with the same sense; and

> thence it follows that, in analyzing such propositions, we must not be guided by Russell's method of symbolizing.

The task of a proper syntactic analysis is to identify the properties that symbolize truth functions. This is done by systematically reducing differences of equivalent formulas resulting in proper representations. These representations are unambiguous not only in the sense that any representation represents one and only one truth function but also in the sense that any truth function is only represented by one and only one proper representation (cf. p. 50 for an explanation of "truth function"). The equivalence procedure we define in part II of this book is such a procedure that converts first-order formulas to unambiguous proper representations.

Any successful realization of this task implies the decidability of first-order logic. According to Wittgenstein's point of view, decidability of first-order logic is a standard of its adequate formal interpretation. This interpretation ensures both (i) that one won't classify what is decidable as undecidable on the basis of misinterpretations and (ii) a decision procedure for the formal properties in question. This is so because converting formulas to a proper notation achieves both: (i) it rules out misinterpreting formal properties as material properties, and (ii) it identifies formal properties by syntactic properties of the notation as a consequence of equivalence transformations. Whatever is true (or false) by virtue of form must be decidable by purely syntactic features of an adequate notation according to Wittgenstein's standards. It is syntactic features that identify the formal properties in question. Formal properties are those properties that are identifiable by pure syntactic manipulation. Thus, it is syntactic manipulation to yield proper symbols identifying formal properties and not derivation from axioms that decides upon formal truths. As long as syntactic properties of a formal representation do not identify formal properties in question one has not established the proper representation of the propositions in question. It is a misconception of a syntactic property to maintain that it is decidable or undecidable as a matter of fact. This conviction is based on proofs that misrepresent propositions concerning form as propositions within a logical formalism concerning facts. Wittgenstein's main contention implies the decidability of first-order logic as a standard of the adequate interpretation of the form of the logical symbolism. In consequence, according to his analysis undecidability proofs rely on an inadequate interpretation of the logical symbolism. These misinterpretations rely on syntactic confusions induced by a logical notation which does not have the adequate syntactic manifold to identify logical properties.

1.4 Concepts of interpretation

In this section, I will explicitly distinguish between different concepts of "interpreting symbols". Before, I specify the kind of symbols we are concerned with. Symbols may be single symbols such as numerals, names, predicates, connectives or they may be complex symbols such as complex arithmetic terms, equations or first-order formulas. However, we are basically concerned with the language of pure first-order logic. This language does neither contain "=" as a logical constant nor function symbols in terms of assigning objects to objects such as "+" in case of natural numbers as domain. Instead, the language of logic we are concerned with comprises as its "logical constants" only logical connectives such as $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ and the quantifiers $\exists, \forall$. In addition, it contains (bound) variables such as x,y,z, brackets and so called "categorematic parts" such as names, predicates and propositional variables. Thus, it is the interpretation of these symbols and the logical formulas composed out of these symbols that is in question in the first place.

Primarily, I distinguish between "interpretation" in terms of identifying the form of symbols, I_F, and "interpretation" in terms of assigning special meanings to symbols, I_M. In respect to I_M, I distinguish between "interpretations" in terms of paraphrases or instantiations on the one side (= I_{MI}) and in terms of assigning "extensions", such as sets or truth values, on the other side (= I_{ME}). I will refer by I_{MI} to paraphrases of logical constants and to instantiations of categorematic parts. Thus, I_{MI} of $\neg$ is "not" and I_{MI} of F may, for example, be "is identical with", "$=$" for short.[7] If I only refer to I_{MI} of the categorematic parts, I will indicate this by "$I_{MI_{cat}}$". In contrast, I will refer by $I_{ME_{cat}}$ to a function, $\Im$, assigning extensions to the categorematic parts. $I_{ME_{cat}}$ corresponds to the use of "interpretation" in standard logic. It assigns a set of objects, the domain $\mathcal{D}$, to variables such as x,y,z. Furthermore, $\Im(t)$ assigns objects from $\mathcal{D}$ to names such as a,b,c. $\Im(\varphi)$ assigns sets of tuples to predicate letters, and $\Im(\mathcal{A})$ assigns truth values to propositional variables. Logical connectives are interpreted$_{ME}$ as functions assigning truth values to truth values and quantifiers are interpreted$_{ME}$ as assigning truth values in respect to $\Im(\varphi)$. Thus, according to I_{ME}, any logical formula A is interpreted as a name of a truth value depending on $I_{ME_{cat}}$. I_{ME} of a logical formula A is a truth value. I will refer to interpretations of formulas of this kind by $I_{ME}(A)$. $I_{ME}(A)$ is either the truth value T or the truth value F. If

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[7] Note, that I do not refer to "=" as a sign of first-order language but as an instantiation of a dyadic predicate F. It should also be noted that this does not presume that this is an admissible instantiation according to Wittgenstein's standards.

$I_{ME}(A)$ is true, $\Im_{ME_{cat}}$ is a model, if $I_{ME}(A)$ is false $\Im_{ME_{cat}}$ is a counter-model. By defining tautologies as formulas being "true relative to all interpretations" one refers to $I_{ME}(A)$ in respect to all $I_{ME_{cat}}$s. In contrast to I_{ME}, I_{MI} allows one to paraphrase logical formulas by sentences of ordinary language. I will refer to such paraphrases by $I_{MI}(A)$. $I_{MI}(A)$s are sentences of a standardized version of ordinary language that correspond one to one to logical formulas. I_{MI} and I_{ME} can be combined by assigning extensions to instantiations, for example, one might instantiate a dyadic predicate F by "=", and define $\Im(=)$ by the class of pairs of identical elements. To any $I_{MI}(A)$ of a logical formula a truth value is assigned according to $I_{ME}(A)$.

In contrast, New Logic is based on nothing but I_F resulting in a formal explication of truth-possibilities of propositions. Such explications determine conditions of truth and falsehood represented by logical formulas by determining how their truth and falsehood depend on primitives of a certain form. I will specify these primitives as "complex poles" (see p. 45). The explications are "formal" explications as they depend on nothing but the features of logical formulas and do not refer to any meaning of propositions instantiating those formulas. They provide a general explanation of the truth conditions of all those propositions instantiating a certain logical formula.

The basic difference of I_F to I_M is that atomic propositional functions are interpreted$_F$ as ab-functions and syntactically treated as propositional functions with two poles. Connectives are then interpreted$_F$ as ab-operations, assigning a- and b-poles to a- and b-poles. Contrary to the syntax of logical formulas, the quantifiers are represented as a pair, one assigned to the a- and the other to the b-pole. This interpretation of the form of the logical symbols results in ab-diagrams of logical formulas. Identifying the symbolizing properties of ab-diagrams then results in a- and b-pole-groups as unambiguous representations of truth-possibilities of propositions. These a- and b-pole-groups represent ab-functions, which, in virtue of their syntactic properties, identify functions of the possibilities of being truth and false of propositions from complex poles. The rules of the ab-notation all serve to convert logical formulas to unambiguous representations of ab-functions. I will give a first outline of this kind of interpretation of the logical symbolism in chapter 2; the whole part II of this book is devoted to give all its details.

In so far as the interpretation of the form of the logical symbolism results in a formal explication of conditions of truth and falsehood, Wittgenstein's interpretation of the logical symbolism is similar to truth-conditions semantics. In fact, accounts of truth-conditions semantics were influenced by Wittgenstein's conception of logic. However, in so far as the formal explication of truth conditions

Wittgenstein envisaged is based on an interpretation of the form of the logical symbolism, which is expressed by the ab-notation and allows for an algorithmetic identification of truth-conditions of propositions representable within first-order logic, it differs from other accounts of truth conditions semantics. However, I do not attribute truth conditions semantics to Old Logic. In particular, mathematical logic, is not based on such semantics. Instead, its semantics is based on I_M.

Figure 1.2 sums up the classification of different concepts of interpretation we are concerned with.

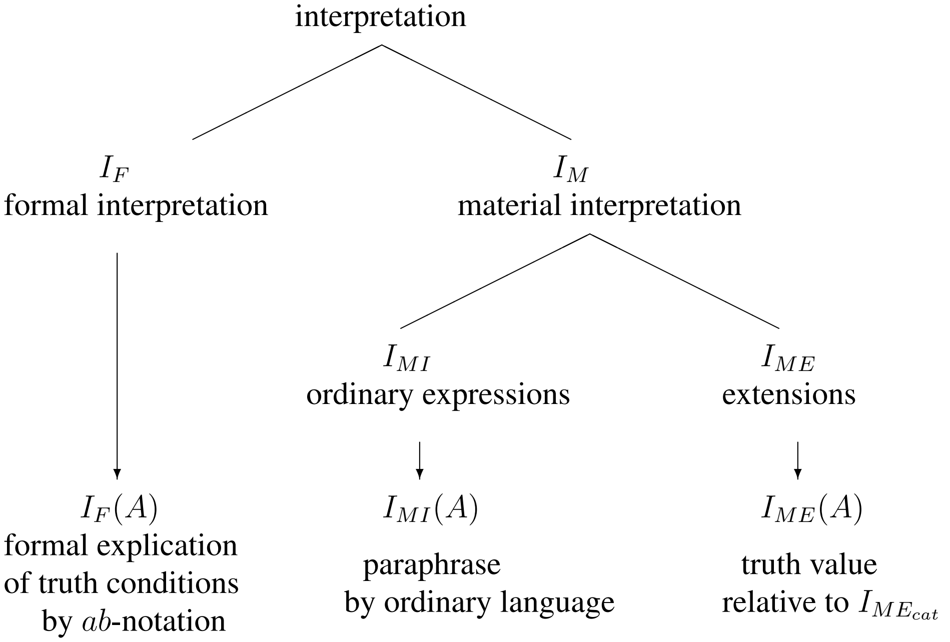

Figure 1.2: Classification of interpretations

1.5 Principles of interpretation

New Logic abstains from I_{MI} as well as from I_{ME}. According to New Logic, any interpretation of the form of a symbolism is prior to any kind of I_M. Before categorematic parts of the logical symbolism are instantiated by certain ordinary expressions and logical formulas are paraphrased by ordinary sentences one must specify the form of possible (admissible) instantiations. To require not more than a grammatical correct paraphrase of logical formulas is too weak to avoid paradoxes and fallacies. Traditional logic postulates the principles of bivalence and extensionality as criteria of adequate $I_{MI}(A)$: any propositions satisfying these two principles are representable within the language of logic. These two principles are, in fact, based on interpreting$_M$ propositions as names of truth values. Thus, traditional logic specifies admissible $I_{MI}(A)$ in respect to $I_{ME}(A)$. According to New Logic, this is too weak to identify formal properties by syntactic means and to avoid fallacies based on the confusion of material and formal properties. In contrast, $I_F(A)$ demands for the principle of bipolarity, the principle of pole-extensionality and the principle of logical independence. These principles are expressed within the ab-notation. New Logic bases $I_{MI}(A)$ on these principles. Due to these principles $I_{MI}(A)$ is by far more restricted than within traditional logic. I explain this in the following.

According to the traditional conception, the *principle of bivalence* is a sufficient criterion to identify those propositions that are representable by atomic propositional variables within logic: Whatever is either true or false (and nothing else) is representable within the language of logic. Within Wittgenstein's conception the principle of bivalence is replaced by the stricter *criterion of bipolarity*. This difference marks the starting point of Wittgenstein's understanding of the language of logic from which all the differences to the traditional conception evolve. The principle of bipolarity states that any atomic proposition has two poles as it is *capable* of being both: true and false. Thus, not the actual truth value of atomic propositions is taken into account but its truth-possibilities. That is why Wittgenstein never refers to truth values of atomic propositions within his conception of logic, but to their "truth-possibilities" (TLP 4.27-4.431). Atomic propositions are restricted to propositions that are materially true or false; they are contingent. This is not based on some metaphysics of modalities or some vague stipulation of a distinction between analytic and synthetic propositions. Instead, this is based on Wittgenstein's main contention that the proper syntax of propositions must make it possible to distinguish between materially and formally true propositions. Bipolarity is introduced as the epitome of what cannot be decided by syntactic means but only by facts. Facts are those entities in reality that make propositions true or false. According to Wittgenstein, the language of logic is first and foremost a proper language to speak about facts. Facts can only correspond to propositions that have the form to allow for facts to decide upon their truth. To be true in virtue of facts (and not in virtue of form) is itself a formal property that must be identified by a syntactic feature of a proper notation. Bipolarity must itself characterize the adequate syntax of propositions symbolizing facts. Thus, within the ab-notation propositional variables are not represented by letters such as P but by structured symbols involving poles such as a-P-b. Likewise, propositional functions are not represented by the form $\varphi(x)$ but by a-$\varphi(x)$-b.

Atomic propositional variables must be interpreted$_{MI}$ by bipolar propositions that are not decidable by syntactic means. Decidability distinguishes between material and formal truth and this distinction becomes manifest in the adequate formal representation. In contrast to the principle of bivalence, the principle of bipolarity rules out $I_{MI}(A)$ in terms of arithmetic propositions. Strictly speaking, it is not $I_F(A)$ that excludes a logical formalization of arithmetic but only $I_F(A)$ in combination with a certain formal analysis of arithmetic expressions. Wittgenstein calls I_{MI} the "application of logic" and insists that logic must neither answer questions of its application nor presume any answers to such questions (cf. TLP 5.557). Thus, logic determines the form of truth-possibilities of propositions but it does neither answer the question nor presume any answer to the question which ordinary propositions fill-out these forms. This, in fact, is the task of a logical analysis of ordinary language which involves analyzing the meaning of ordinary expressions. Certainly, this task is cardinal for philosophy and its business to clarify the syntax of propositions. However, in order to perform this task the syntax of meaningful propositions must be specified first, which is done within logic. This, in turn, brings about that any application of logic can't help but determining truth-possibilities. The question for a proper syntactic analysis of ordinary propositions is whether these truth-possibilities indeed capture the use of the propositions in question. Thus, if one interprets$_F$ the formula P by a-P-b and interprets$_{MI}$ P by "1 is identical with 1" one must draw the consequence that this proposition has meaningful conditions of being false. However, this, in fact, does not agree with its actual use, which is captured by a formal interpretation, I_F, of arithmetic propositions based on the understanding of numbers as forms and identity of numbers as a formal relation.

New Logic is not only independent of I_{MI} but also of I_{ME}. It neither determines nor presumes the truth value of any proposition or the extension of any predicate. Due to $I_F(A)$ one can abstain from interpreting$_{ME}$ logical formulas by assigning to them truth values of their instances. Wittgenstein's conception allows him to dispense with both, (i) truth values in terms of objects that are referred to by propositions and (ii) a function of interpretation assigning objects to atomic propositions and extensions to predicates. Indeed, within his conception there is no need to rely on any entity that meaningful propositions *refer* to as meaningful propositions do not refer to anything. Instead, they state the existence of facts; they leave open to reality to decide upon their truth value (cf. TLP 4.463). Formally true propositions are not meaningful; they are proven by formal means, whereas truth values of meaningful propositions are a matter of fact. As such they cannot be decided by logical means nor must any decision about logical properties depend on the truth or falsehood of any proposition. In consequence, logic must not be based on axioms nor can logic prove any theorem in terms of propositions with a certain meaning. Determining the formal conditions of *possibilities* of being true or false is prior to assigning truth values to certain meaningful propositions.

In addition to the principle of bivalence traditional semantics is based on the *principle of extensionality*. The principle of extensionality says that the truth value of a complex proposition depends on nothing but the truth values of its parts. Thus, sentential connectives such as $\wedge, \vee, \rightarrow, \leftrightarrow$ are interpreted$_{ME}$ as *truth functions*, assigning truth values to truth values. New Logic replaces the principle of bivalence with the principle of bipolarity. In consequence of this latter principle and the interpretation$_F$ of connectives in terms of ab-operations, the traditional principle of extensionality is replaced by the principle that propositions occur within propositions only as the bases of *truth operations* (TLP 5.54). Thus, $\wedge, \vee, \rightarrow, \leftrightarrow$ are interpreted$_F$ by the ab-notation as truth-operations assigning in any case *two* poles to *all possible* pairs of poles. We label the first, traditional principle "the principle of *truth-value extensionality*" and the second "the principle of *pole extensionality*". Thus, according to traditional logic, logical formulas represent the dependence of the *actual* truth value of *different* propositions instantiating the logical formula on the actual truth value of their respective parts. In contrast, according to New Logic, logical formulas represent the dependence of the *possibilities* of being true and false of *one* proposition instantiating the logical formula on the truth-possibilities of its parts. Not different propositions correspond to different truth values but different truth-possibilities correspond to one proposition. Thus, for instance, consider "Berlin is the capital of France or Washington is the capital of Great Britain" as an instantiation of $P \vee Q$. According to traditional semantics, I_{ME} is the truth value FALSE as both disjuncts are, in fact, false. In case of other instantiations of the same formula, e.g. "Berlin is the capital of Germany or Washington is the capital of Great Britain", I_{ME} is TRUE. New Logic, in contrast, does nothing but explain truth condition of admissible instantiations: Thus, both of the mentioned instantiation of $P \vee Q$ are true in case at least one of the disjuncts is true and false otherwise. One does not refer to the actual truth-value of the whole proposition or its parts in the explanation of the truth conditions; all one refers to are different truth-possibilities of the complex proposition as well as its parts.

Whether or not the principle of pole-extensionality rules out any application of logical formulas to propositions on so-called propositional attitudes such as "Peter believes that Berlin is the capital of France" depends on the logical analysis of those propositions. In TLP 5.54-5.5423 Wittgenstein argues against an analysis of such propositions in terms of stating a relation between a subject and a proposition. Such an analysis would be in conflict with both of the mentioned principles of extensionality. Wittgenstein, once more, accuses such an analysis to confuse grammatical form and logical form. Yet, New Logic is independent of the question of the proper analysis of propositions concerning propositional attitudes. All that it lays down is that *if* those propositions are representable within the language of first-order logic *then* they must be analyzed as propositions depending on nothing but the truth-possibilities of primitive propositions. This even holds true in case of propositions on modalities, which are non-sensical pseudo-propositions according to Wittgenstein's analysis (cf. TLP 5.525). Of course, New Logic is highly motivated by identifying such propositions as pseudo-propositions and it seems to be quite absurd according to the standards of New Logic to represent such propositions *within* the language of logic (or some expansion of it). Yet, strictly speaking, New Logic does not imply an analysis of ordinary proposition about modalities, it only provides means to identify contingent, logically necessary and logically impossible propositions within the language of logic. Yet, if one insists to analyze "It is logically necessary that it rains or it does not rain" or "It is necessary that 1 + 1 = 2" as proper propositions instead of analyzing them as pseudo-propositions this does not conflict to the conception of New Logic but to Wittgenstein's analysis of such ordinary propositions. New Logic does not depend on representing ordinary propositions within the language of logic; yet it does depend on the presumption that any adequate logical formalism must formally identify truth-possibilities and thus be decidable.

According to the principle of pole extensionality the truth-possibilities of complex propositions depend on nothing but the truth-possibilities of its parts. Thus, the truth-possibilities of any proposition depend on the truth-possibilities of ultimate, unanalyzable parts of any propositions. According to TLP, these propositions are atomic propositions.[8] The *principle of logical independence* says that the possibilities of being true or false of any atomic proposition is independent of the possibilities of being true or false of any other atomic propositions (cf.

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[8] I will replace the concept of atomic propositions by the broader concept of complex poles that applies to first-order logic (cf. p. **??**. In contrast to atomic propositions, complex poles are not necessarily logical independent. Yet, it remains true that any ab-operation assigns an a- and a b-pole to all of the four combinatory possible combinations of poles. This is the essence of the principle of logical independence. Furthermore, it is to be noted that any elimination of combinations of poles in later steps of the ab-notation is due to purely formal relations between poles identified within the ab-notation.

TLP 4.27f., 4.42, 4.211, 5.135, 2.061f). This implies that combining propositions to complex propositions does not restrict the space of possible combinations of truth-possibilities. Within the ab-notation this is expressed by the fact that any ab-operation, representing dyadic connectives such as $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ assign poles to *all* of the four combinations aa, ab, ba, bb. There is no equivalent to the principle of independence within traditional logic as this principle essentially rests on considering truth-possibilities instead of truth-values. The three principles underlying I_F are expressed within the ab-notation by the way how the outmost poles are determined by inner poles.

On the basis of a semantics explicating truth conditions of propositions, the principle of logical independence is a very strong constraint for $I_{MI}(A)$. For example, given that it is impossible for two colors to be at the same place at the same time, it rules out to represent "Red is here now and Blue is here now" as a conjunction of two atomic propositions (cf. TLP 6.3751). The same holds true in general for combinations of propositions stating gradual properties (e.g. "This particle is faster than that particle") and containing numbers (e.g. "The temperature in this room is 20° celsius.", cf. RLF, p. 167f.). This does not necessarily mean that those propositions are not representable within first-order logic. It only means that those propositions must be analyzed further into propositions that satisfy the principle of logical independence in order to be representable within the language of logic. In TLP, Wittgenstein indeed believed in such an analysis (cf. in detail Lampert (2000), chapter 4). However, later he abandoned to it (cf. RLF). This was the reason for Wittgenstein to dispense with his early belief that the language of first-order logic serves as a universal language for speaking about facts. According to I_F, expressing ordinary propositions within the logical formalism either (i) involves rather sophisticated methods of analyzing ordinary propositions or (ii) is impossible even in case of paradigms of primitive, empirical propositions. Whereas Wittgenstein believed in (i) in TLP he later shifted more and more towards (ii).

Of course, the *motivation* of Wittgenstein's conception of New Logic was to represent *all* meaningful propositions as well as all their possible formal dependencies within the language of logic. In TLP, he was convinced that this indeed was possible. Yet, this conviction highly depends on his Tractarian conception of analyzing meaningful propositions. However, New Logic does not depend on this conception of analysis. Rejecting this conception of analysis does not reject $I_F(A)$, it only diminishes the application of logic to ordinary propositions. The language of logic is appropriate, "but only for" a "narrowly circumscribed region, not for the whole of what" Wittgenstein was originally "claiming to describe"

(PI §3). In so far one claims for a formal representation of ordinary propositions additional notations are necessary.

Criticizing that logic serves as the universal language to speak about truth is already in the line of TLP. Compared to the tradition of mathematical logic, Wittgenstein already diminished the application of logic in TLP as he rejected to apply it to arithmetic. This is primarily based on the principle of bipolarity, which, in turn, is grounded in his intention to distinguish between formal and material properties and to restrict the logical calculus to meaningful propositions. To identify formal properties and relations of arithmetic special arithmetic notations are required. Wittgenstein's later criticism of the Tractarian conception of analyzing meaningful propositions enforced him to go further in diminishing the application of logic and to abandon the conviction that logic serves as the universal language to speak about facts. This criticism is primarily based on the principle of logical independence. However, later on, it dilates to the abandonment of the language of logic as a significant tool to analyze ordinary propositions. Whereas traditional logic intends to use the language of logic as the basic language of truth; the early Wittgenstein diminishes this claim and restricts the application to meaningful propositions; the middle Wittgenstein then went on in diminishing the application of logic by recognizing that not all meaningful propositions are expressible within a language of logic based on his three principles. Later on, he then rejected that the language of first-order logic were a useful tool to come to understand ordinary propositions; it is rather an ideal that cannot capture the complexity and vagueness of ordinary language (cf. PI, §96-108). However, this criticism of the applicability of logic is first and foremost based on analyzing ordinary propositions and not on analyzing first-order language. "Logic must look after itself" (TLP 5.473) as determining the form of symbols is prior to giving meaning to them. Logic is concerned with the *possibility* to express sense according to certain formal conditions, it is not affected by the range of propositions satisfying these conditions. Therefore, diminishing the applicability of first-order logic does not concern its syntactic analysis.

The ultimative standard of Wittgenstein's theory of representing propositions within the logical formalism is not expressive power or the ideal of an effective procedure to represent ordinary propositions by use of logical formulas. Instead, first and foremost the logical formalism must make it possible to distinguish materially true (or false) propositions from formally true (or false) propositions in respect to all those propositions representable within the logical formalism (whatever these propositions are). This presumes the decidability of the formalism as formal properties are those that are decidable by converting the formulas to a proper notation identifying the properties in question. Decidability is a matter of the adequate syntactic analysis and formal representation of the formalism. Any proper formal language must be decidable in the first place in order to adequately represent formal properties. This determines what is expressible within the formalism and not the other way round. The question of logic is not which proposition are of a certain logical form but to define logical forms properly. New Logic defines a logical system that identifies truth-possibilities. The correct $I_F(A)$ must rule out any misinterpretation based on confusing material and formal properties before it allows for I_{MI}. In particular, only this guarantees that one won't represent pseudo-propositions as meaningful propositions and classify what is decidable as undecidable. Thus, the intention of New Logic is not to establish logic as a basic language of science. Instead, its intention is to specify a proper understanding of the formal properties of logic.

1.6 Restricted interpretations and Church's theorem

On the one hand, interpreting first-order logic on the basis of the three mentioned principles provides new options to decide logical properties by referring to syntactic properties of a notation based on these principles. On the other hand, this way of interpreting first-order logic restricts applying the language of logic. If one comes to see both of these sides, undecidability of first-order logic becomes questionable. One might achieve decidability for the price of restricting admissible I_{MI} in consequence of replacing the principles of classical semantics by principles that are more restrictive and express themselves in a proper logical notation. As this rules out applying logic to arithmetic undecidability proofs are rejected by New Logic. Thus, for example, it by no means follows from the decidability of first-order logic that the halting problem for Turing-machines is solvable. Instead, it follows that Turing-machines are not adequately represented within the language of first-order logic. Like other so called "undecidable problems" the halting problem is ill-defined as it confuses pseudo-propositions with real propositions, which in turn is grounded in not distinguishing syntactically between "computable functions" and propositional functions. Computability is itself a standard of a proper arithmetic syntax. Likewise, it does not follow from the decidability of first-order logic that mathematics "ceases to exist" as all mathematical problems would now become solvable by mechanical means (cf. Neumann (1927), p. 11f.). Instead, it follows that one should not apply first-order logic to arithmetic. Rather, one should invent mathematical notations to decide mathematical problems. There is no reason to assume that all mathematical problems were solvable within one notation; mathematical ingeniouity goes hand in hand with inventing new notations and decision procedures. If a "mathematical" problem is not decidable it is either no mathematical problem or one has not yet invented the proper syntax to deal with it. Solutions of mathematical problems call for decision procedures; mathematics "ceases to exist" as soon as its problems become undecidable.

From the point of view of New Logic the question is not how it can be possible what is commonly judged to be impossible. The question is rather whether it is indeed the same what is in question. Is it possible to refute Church's theorem on the basis of a syntactic analysis of first-order logic that is in conflict with traditional semantics? In particular, two questions arise: (i) Is the concept of decidability the same? (ii) Is the concept of being a tautology the same? The answer to both of these questions is to the negative. Nevertheless, it still follows from New Logic that Church's theorem is refuted. I am going to explain this in the following.

Traditional logic specifies the concept of decidability by the concept of computable characteristic functions, which are, in turn, identified with primitive recursive (or Turing computable) characteristic functions. The key point is that one refers to the extensional concept of a function; the distinction between what is computable and what is not computable is drawn within the realm of functions. Functions assign values to objects that are referred to by names. Whether or not an entity has a certain decidable property is not seen as an internal property of its form. According to Wittgenstein's view the distinction between what is computable and what is not computable is a distinction of categories. It is not a distinction between different kinds of functions. Instead, Wittgenstein restricts the concept of a function to what is not decidable and analyzes computability by his concept of an operation. Operations do not assign values to objects according to some external criterion. Instead, they systematically vary properties of forms. Their inputs (bases) and outputs (results) are forms and the operation is a syntactic device applied mechanically to vary those forms. Decidability means computability of a syntactic property by manipulating forms such that the resulting form identifies the property in question. It is nothing but equivalence transformation terminating within a finite number of steps that decides whether a certain form has a property or not.

As we will see, this understanding of computability is even more restricted than the traditional conception as it does not allow for deciding a property due to an external criterion; the properties in question must be decidable due to syntactic, internal criteria of a proper representation. Mathematical operations are expressible by mathematical formulas. Any mathematical formula that expresses an operation also expresses a "computable function" in terms of its traditional understanding (but not the other way round). Wittgenstein rejects to analyze those mathematical operations by logical means and to represent them as functions within a logical formalism. His rejection of a wide, extensional concept of functions that comprises computable as well as uncomputable functions motivates his denial of undecidability proofs. Instead, he calls for a syntactic distinction between what is computable and what is not computable and draws this distinction by distinguishing iterative applicable operations, which vary forms by syntactic manipulation and functions, which assign values to their arguments due to external criteria. The question as to the decidability of a formal property calls for inventing a proper syntax based on a proper syntactic analysis. It cannot itself be represented adequately within some formalism in terms of stipulating a characteristic function and asking whether this function is primitive recursive or not. Wittgenstein's analysis of computability calls for a syntactic distinction of what is computable by syntactic means and what is not determined by properties of symbols and it rules out to treat questions concerning the computability of formal properties as proper questions that have a well-defined meaning independent of answering them. As long as one has not specified how to compute formal properties their definition remains insufficient.

For now it suffices to note that (i) New Logic envisages an analysis of computability and decidability that significantly deviates from that hold within modern mathematical logic and (ii) it still holds that whatever is computable and decidable according to New Logic is a forteriori computable and decidable according to traditional logic. Thus, a decision procedure according to New Logic is a decision procedure according to traditional logic. It is a decision procedure in the most intuitive sense of a purely mechanical advice manipulating nothing but syntax of symbols according to an unambiguous and definite strategy. It can be carried out by a computer within a finite number of steps. In particular, the procedure I specify for first-order logic definitely is a decision procedure as it is nothing but an equivalence transformation according to well known syntactic rules that terminates in a finite number of steps. Moreover, it is already implemented by a rather trivial computer program within *Mathematica*.

However, one might doubt whether such a decision procedure indeed decides the property of being a tautology in terms of its traditional understanding. First of all, it must be conceded that the concepts of being a tautology are different in Old and New Logic. According to New Logic a logically true formula is not defined as a formula, to which $I_{ME}(A)$ assigns the value TRUE relative to all $I_{ME_{cat}}$s. Instead, a tautology is defined as a formula of a form that does not provide for the possibility of falsehood. As we will see, truth-possibilities of formulas are determined by a- and b-pole-groups within the ab-notation. Thus, logically true formulas are those that, represented within the ab-notation, lack consistent b-pole-groups, which itself will be defined on the basis of nothing but syntactic properties. Tautologies are propositions of this form. Being a tautology is a possible form of single propositions and it is defined independently of any propositions filling out this form. Thus, one need not refer to several (or an infinite or even a "transfinite" number of) propositions and their truth values to define logically true formulas. Instead, it is nothing but a syntactic property, which defines logically true formulas and nothing but finite syntactic manipulation that identifies them. According to New Logic the traditional conception of a tautology within first-order logic is ill-defined as it goes beyond considering formal properties and refers to a problematic totatility of $I_{ME_{cat}}$s.

However, the difference of the mentioned definitions of logically true formulas does not bring about that different formulas are said to be logically true. First of all, this is due to the fact that the decision procedure I define in part II of this book can be specified as an *equivalence procedure* according to well-known equivalence rules. Thus, in fact, I specify a procedure that converts any first-order formula to an unambiguous representative within a finite number of steps. Thus, tautologies are identified by converting their negations to disjunctions of explicit contradictions, which in turn is equivalent to $\perp$. Thus, as we use nothing but as accepted equivalence rules, the original formula is equivalent with $\top$. In contrast, if the original formula is not equivalent with $\top$ its representative will not be $\top$. Thus, it suffices to refer to a purely syntactic account of being a tautology, which amounts to be equivalent with some theorem, say $P \vee \neg P$.

Furthermore, I will specify rules how to construct a counter-model from representatives of non-tautologies. In general, I will specify how to enumerate what I will call "lawful" models and counter-models of a logical formula given its representative within the ab-notation. Those representatives serve as identity criteria of those kind of models and counter-models as they reveal the structure of those models and counter-models by their syntactic properties. According to traditional semantics "lawful" models and counter-models are a subclass of the totality of models and counter-models. This is for the reason that lawful models and counter-models are restricted to models with an enumerable domain and with enumerable "extensions" of predicates. However, according to New Logic those lawful models and counter-models specify *all possible extensions* distinguished in those that satisfy conditions of truth and those that satisfy conditions of falsehood. In case of tautologies, the class of models simply coincides with all possible extensions, whereas the class of counter-models is empty. Whereas Old Logic associates different propositions with different models (or counter-models), New Logic associates different structures of models (and counter-models) with non-equivalent formulas. Yet, the class of models and the class of counter-models of a logical formula remains the same in so far only "lawful" models and counter-models are considered. New Logic restricts the realm of possible extensions in order to rule out misconceptions of truth possibilities according to a proper syntactic analysis.

New Logic claims that both, the realm of possible I_{MI} as well as the realm of possible I_{ME} are not sufficiently restricted within traditional logic. This induces Church's theorem. Because one makes it possible to represent what, in fact, is impossible to represent according to a proper syntactic analysis, it becomes impossible what, in fact, is possible according to a proper syntactic analysis. Church's theorem becomes obsolete as soon as a proper syntactic analysis of first-order logic serves for both: (i) identifying possibilities of being true and false and thus (ii) making it impossible to represent nonsense within the logical formalism.

Chapter 2

First-order logic

In this chapter we give a first outline of the syntactic analysis underlying the ab-notation. As we intend to explain the general ideas rather than technical details, we only present basic examples. It is the task of part II of this book to show how the ab-notation applies to the whole realm of pure first-order logic.

2.1 Introduction

In his earliest writings, *Notes on Logic* (NL)[1], *Moore Notes* (MN) and *Tractatus logico-philosophicus* (TLP)[2], Wittgenstein compares his conception of logic to "Old Logic". By "Old Logic" he means the logic of Frege and Russell. In contrast, his own conception of logic stands for a "New Logic". From 1912 to 1914, he confronted Russell with his work on New Logic. Soon, Russell accepted Wittgenstein as his "master" (cf. Monk (1990), chapter 3). He wanted Wittgenstein to work on a third edition of *Principia Mathematica* (PM).[3] Finally, Russell expected that the elaboration of Wittgenstein's New Logic would displace PM as the paradigm of modern logic. However, this expectation was not fulfilled. The common explanation for this is that Wittgenstein's conception of logic could only be realized in propositional logic and not in predicate logic.

This judgement is well justified. The crucial difference between Wittgen-

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[1] Despite the editorial deficiencies of the 1979 edition, I will refer to it because it is used frequently. If necessary, the critical edition Biggs (1996) is taken into account.

[2] I refer to the familiar translation of Pears and McGuinness.

[3] cf. Pinsent (1990), p. 60, diary entry from 29th August 1913, see also p. 37, entry from 25th October 1912.

stein's conception of logic and Frege's and Russell's conception is that Wittgenstein hoped to solve logical problems by "finding a form of representation in which all and only logical equivalents have exactly one and the same expression" (Landini (2007), p. 112.). In virtue of an ideal notation to which predicate formulae must be converted he intended to give a clear account of what is meant by saying that logical propositions are "true in virtue of their form."[4] Whereas the syntactic properties of predicate formulae do not unambiguously represent their truth conditions, an ideal symbolism must satisfy this criterion. Thus, it allows one to identify the truth conditions of the initial predicate formulae and their internal relations to other formulae by the mere syntactic properties of their ideal representation. From this point of view, neither a theory of deduction nor model theory are the distinct tools to solve logical problems. Furthermore, the analysis of meta-logical concepts such as "... is a theorem" or "... is a tautology" as real concepts with content is obsolete. Instead, what counts is a mechanical procedure to convert predicate formulae to their ideal representation, which reveals the logical properties of the initial formulae. This is not only in conflict with Frege's and Russell's conception of logic but also stands in stark contrast to modern mathematical logic. This conflict culminates in Church's theorem that predicate logic, contrary to propositional logic, is undecidable. As rightly noted by a number of experts, Wittgenstein's conception of logic implies its decidability. Thus, it seems, that his outlook of logic is refuted by modern mathematical logic because it is proven to be not realizable.[5] In fact, no suggestions of proofs of predicate logic can be found in TLP. This seems to confirm the common judgement that Wittgenstein's conception of logic cannot supply any substantial contribution beyond propositional logic.

However, before brushing aside Wittgenstein's approach on the basis of modern logic one should specify the motives and fundamental notions of his alternative as well as take seriously the attempt to realize it in predicate logic. Wittgenstein's New Logic does not only imply a critique of the traditional proof conception in predicate logic but also of modern metamathematical proof methods. The method of diagonalization, Church's thesis, the understanding of logical concepts in terms

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[4] cf. Russell (1937), p. xii who confessed to be unable to do this.

[5] cf. Landini (2007), p. 118: "The undecidability of quantification theory is a significant blow to Wittgenstein's conception of logic. [...] it undermines Wittgenstein's hope of finding a notation in which all and only logical equivalents have one and the same representation." cf. also Black (1964), p. 323, Anscombe (1996), p. 137, Sundholm (1990), p. 60, Floyd (2005), p. 95. This judgement is even prominent in the editor's comment upon Wittgenstein's writings, cf. the footnote of the editors in *Cambridge Letters* (CL), p. 52.

of propositional functions of meta-language are all part of a paradigm that is rejected on the basis of Wittgenstein's way of a purely syntactical foundation of logic and arithmetic. Thus, before rejecting Wittgenstein's conception on the basis of Church's theorem, one must consider his criticism of metamathematical proof methods.

Furthermore, critics often fail to take into account that Wittgenstein did not think of truth-tables as the proof method of his New Logic. Instead, he invented his so-called "ab-notation" for this purpose. The ab-notation is a logical notation on which he worked intensively from 1913 to 1914. It is this notation he identifies with the "new notation" in opposition to the "old notation" of Frege and Russell (NL, p. 93[1]). The use of truth-tables – "WF-schemata" in Wittgenstein's terminology – is not peculiar for his logic conception.[6]

It is a misjudgement that the method of truth-tables displaced the ab-notation. Wittgenstein already used them in 1912 before he worked on the ab-notation 1913/14 (cf. figure 2.1, quoted from Shosky (1997), p. 20). Contrary to the method of truth-tables, Wittgenstein's intention by developing the ab-notation was to realize his conception of logic in the realm of predicate logic (cf. CL, letter 28, p. 4, against Biggs (1996), p. 27). Unlike truth tables, the ab-notation essentially involves using of quantifiers. The question of how far Wittgenstein's New Logic can be realized depends first and foremost on the question of how much his ab-notation is applicable to predicate logic.

Unfortunately, the notebooks from 1913 to 1914 dealing with the ab-notation were never found (cf. CL, letter 32, p. 58 and Biggs (1996), p. 11). Thus, one must rely on the scanty remarks in NL, MN, and CL from 1913 and 1914. In addition, understanding of the ab-notation, even in the realm of propositional logic, was hampered by the fact that all received diagrams of the ab-notation were either reproduced with errors or not even printed in the first editions of NL, MN, and CL (cf. the comments on the diagrams on p. 123, p. 125 and p. 132). Furthermore, Wittgenstein does not refer to the ab-notation as a proof method for predicate logic in TLP. Instead, he presumes that quantified propositions are reducible to truth functions of an infinite number of atomic propositions; a thesis he later branded his "biggest mistake of the TLP" (Wright (1982), p. 151). This all accounts for Wittgenstein's ab-notation remaining nearly disregarded in the literature until now. However, Wittgenstein did not doubt the validity of the ab-notation for the whole realm of predicate logic. Not the possibility to apply his ab-

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[6] cf. also Landini (2007), p. 118-124 who argues for the thesis that "truth tables were not Wittgenstein's innovation".

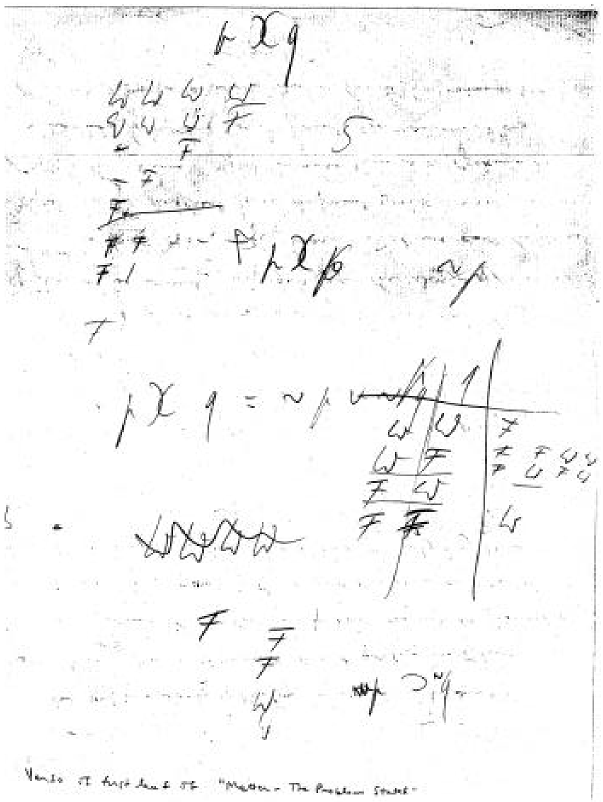

Figure 2.1: reverse side of Russell's manuscript "On Matter" with sketchy truth-tables handwritten by Wittgenstein from 1912

notation to predicate logic but only the handling of identity within the ab-notation was an open question for him (cf. CL, letter 30, p. 53). Likewise, he never confines his understanding of logical proofs to propositional logic. He still speaks of the "Old Logic" (TLP 4.126 using "alte Logik" in the German original as well as TLP 6.125 and NB p. 89, cf. also CL, p. 126 for "old notation") which implies a "New Logic" as its opposite. It was not Wittgenstein's intention to work out in detail his conception of a New Logic on the basis of the ab-notation, but he had no doubt of the feasibility of this project.

The work at hand attempts to work out Wittgenstein's idea of a New Logic for predicate logic (without identity[7]). To do this, a first step is to establish a general

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[7] Although Wittgenstein's theory of identity is one of the most significant and radical features of his conception of a New Logic, we will not discuss this topic. Considering it, would require

characterization of Wittgenstein's programme of a New Logic (chapter 3). As we will see, this involves a critique of the axiomatic proof conception as well as a rejection of Church's theorem and its underlying notions and proof methods. The general characterization of New Logic also provides the background on which to interpret the particular remarks on the ab-notation in Wittgenstein's earliest writings (chapter 4). This, again, will be the starting point to systematically elaborate on his programme in Part II. A systematic elaboration shall make it possible to judge the historical and systematic significance of Wittgenstein's New Logic, which pretends to be an alternative to Frege's and Russell's classical conception of logic. The systematic elaboration of Wittgenstein's logic will consist of two parts. In chapter 5, the ab-notation will be defined for a certain subclass of predicate logic, one we label "elementary predicate logic". Formulae of elementary predicate logic are only those predicate formulae that do not have dyadic connectives, such as $\wedge, \vee$ or $\rightarrow$, in the scope of quantifiers. By referring to elementary predicate logic, we can clarify the principles of expanding the ab-notation from propositional logic to predicate logic, and show how Wittgenstein's conception can be realized within a manageable portion of predicate logic. In chapter 6, Wittgenstein's program will then be carried forward to the whole realm of pure predicate logic. It will be shown how to convert predicate formulae to a form of representation in which all and only logical equivalents have exactly one and the same expression. Furthermore, we will show on this basis (i) how to explain the conditions of truth and falsehood of predicate formulae and (ii) how to specify the totality of internal relations between predicate formulae. Thus, we intend to demonstrate that Wittgenstein's conception of logic is not a philosopher's dream disproved by mathematical methods but a realizable logical programme disproving modern metamathematical proof methods.

To be sure, Gödel's incompleteness proof is certainly more admirable from the point of intellectual ingenuity than its puristic linguistic critique. Yet, this does not release it from such a critique. Sciences, mathematics in particular, is not simply an enterprise to describe and explain facts but also an engagement to specify an adequate syntax to do so. The use of a certain syntax determines a system of concepts, techniques, problems and paradigmatic solutions that determine proofs within this system. The belief that theorems are beyond doubt that are proven within a certain system and acknowledged by the community work-

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working out a theory of exclusive predicate logic. According to our understanding, this forces radical changes in the semantics of predicate logic. Instead, we want to elaborate Wittgenstein's conception for predicate logic with its common inclusive interpretation and semantics.

ing within this system is due to a naive understanding of science. The aim of Wittgenstein's philosophical analysis is not to contribute to scientific systems but to clarify their conceptual presumptions. Wittgenstein compares the influence of this enterprise to the increase of mathematics with the influence of sunshine to potato shots, which grow metres long in the darkness (PG II, §25). Wittgenstein was rather interested in the philosophical critique of mathematical logic than the concrete elaboration of an unmistaken syntax. That is the main reason why he himself did not elaborate his conception of New Logic in detail. Yet, he was sure that his philosophical method was capable of having a strong influence on science. He was convinced that the "mathematician of the future" would be distinguished by a higher sensitivity for the kind of syntactical analysis of language Wittgenstein envisaged (PG II, §25). His philosophy of logic and mathematics is meant to influence logic and mathematics. Yet, this influence does not consist in a contribution within established logic or mathematics. It rather consists in proposing a different form of representation within logic and mathematics. Due to this form of representation the concepts of logic and mathematics are clarified. On the one hand, this makes problems solvable that are not solvable by traditional means. On the other hand, this shows that a huge amount of modern mathematical logic is rather a methodical proliferation of conceptual confusions than a significant progress of knowledge. Thus, for example, according to Wittgenstein's point of view solving the decision problem of logic by no means helps solving mathematical problems. This is for the simple reason that mathematics is not represented adequately within logic. Thus, the expected influence is rather to abandon the hope to solve mathematical problems by mechanical means within logic than to make it possible to solve non-trivial mathematical problems by a mechanical procedure.[8] Unsolved, non-trivial mathematical problems must be solved by inventing hitherto not recognized syntactic relations (PG II, §25). They are not solved by simple applying the rules of a known calculus to an already clear representation of the problem. Instead, the solution of the problem involves the problem of the adequate syntactic representation of the problem in question. Thus, it cannot be said to be decidable or undecidable in advance as the problem is not representable adequately without solving it. It is once more the prosaic form of posing a problem within ordinary language that suggests that a problem seems to be clear and solvable without specifying new syntactic rules (cf. PR, §159).

Wittgenstein focussed on this second kind of contribution rather than on demon-

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[8] cf. Neumann (1927) who, as many others, thought that "mathematics would cease to exist" if logic would be decidable.

strating the superiority of his approach (cf. PG II, §25).

However, it was hardly ever recognized that Wittgenstein's perspective of a New Logic articulates a substantial alternative to modern mathematical logic. This has several reasons. In the following, we discuss five of them: (i) Wittgenstein refused to elaborate his conception of logic, (ii) mathematical logic is a well established discipline, which seems to be beyond doubt, (iii) Wittgenstein's conception of logic implies its decidability, which is incompatible with Church's theorem, (iv) it is not recognized that the "new notation" of New Logic is Wittgenstein's ab-notation rather than truth tables, (v) it is maintained that Wittgenstein himself rejected to his original project of a New Logic or that he never meant to criticize mathematical logic as a philosopher.

(i) Wittgenstein refused to elaborate his conception of logic in detail. Wittgenstein's contemporaries in Cambridge as well as in Vienna did expect a considerable contribution to logic and the foundations of mathematics from him. From 1912 to 1914, Wittgenstein confronted Russell with his work on New Logic. Soon, Russell accepted Wittgenstein as his "master" (cf. Monk (1990), chapter 3). He wanted Wittgenstein to work on a third edition of *Principia Mathematica* (PM).[9] Finally, in this period Russell even thought that the elaboration of Wittgenstein's New Logic might displace PM as paradigm of modern logic. However, his sympathy was cooled down by the fact that Wittgenstein did not present a systematic account of his point of view. Likewise, the Vienna circle praised Wittgenstein for being the first one who was able to explain what it means to be "true in virtue of form" in TLP. However, this explanation remained without significance for logic as it remained unclear how to apply it beyond propositional logic. On the 2. congress of epistemology of the exact sciences in Königsberg 1930 Wittgenstein's approach was presented by Waismann as an alternative to logicism, formalism and intuitionism that were presented by Carnap, Neumann and Heyting. However, it remained rather undiscussed as it seemed not be "ripe of decision" (Hahn (1931), p. 141). Although Wittgenstein's ideas had some influence on the philosophy of logic they did not have a lasting influence on logic or the foundations of mathematics because he refused to present a logical elaboration of his point of view. He was interested in the philosophical foundation of his programme rather than in its logical realization. Instead of speculating in how far Wittgenstein was able or not to systematically elaborate his point of view, we take the effort to demonstrate

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[9] cf. Pinsent (1990), p. 60, diary entry from 29th August 1913, see also p. 37, entry from 25th October 1912.

how his conception is realizable within pure first-order logic in this book. We hereby also show that his philosophy is not separable from questions of logic and the foundations of mathematics.

Wittgenstein's philosophy is meant to influence logic and mathematics. Yet, this influence does not consist in a contribution within established logic or mathematics. It rather consists in proposing a different form of representation within logic and mathematics. Due to this form of representation the concepts of logic and mathematics are clarified. On the one hand, this makes problems solvable that are not solvable by traditional means. On the other hand, this shows that a huge amount of modern mathematical logic is rather a methodical proliferation of conceptual confusions than a significant progress of knowledge. Wittgenstein focussed on this second kind of contribution rather than on demonstrating the superiority of his approach (cf. PG II, §25).

Another reason for disregarding Wittgenstein's New Logic lies in the fact that hardly any scientific discipline is as established as mathematical logic. Every student of mathematics or philosophy comes to know predicate logic in its standard version. A huge variety of logic-textbooks exist. Besides differences in details, syntax, semantics and meta-logical properties are rather uncontroversial. The notions of a well-formed formula, of a logical proof or of models / counter-models are defined rigorously. Precise strategies are specified to represent arithmetic or meta-logical propositions within the language of logic. This makes it possible to answer questions as to the possibilities and limits of logic and arithmetic in an exact mathematical manner. Church's theorem of the undecidability of predicate logic or Gödel's incompleteness theorem of arithmetic mark the corner-stones of any investigation in logic or arithmetic as well as for any philosophy of logic and arithmetic. Logic is used as the fundamental language of mathematics and the basic concepts, methods and results of mathematical logic seem to be beyond any question.

Thus, from a mathematical attitude to logic it seems rather strange to consider an alternative. However, this attitude results from the constraints of a scientific paradigm. We will argue why this paradigm is deficient. Yet, thinking within the tradition of mathematical logic makes it almost impossible to understand Wittgenstein's conception of logic. Mathematicians are mostly alienated by Wittgenstein's remarks on the foundations of mathematics. Philosophers with more sympathy for Wittgenstein's point of view are still concerned with the question in how far his philosophy of mathematics is related to the rise of mathematical logic. However, to do justice to Wittgenstein's approach one must go back to the time before mathematical logic was established and recognize the kind of problems Wittgenstein was concerned with. It was not before the second half of the 19th century where logic and mathematics got into contact. This involved two radical changes: First of all, the symbolic language of algebra was used to formalize logical proofs and second, logic was used to reformulate mathematical proofs.[10] This culminated in the predicate calculus as the basic language of logic and arithmetic. Finally, the axiomatic proof systems of set theory should capture mathematics. The foundations of mathematics changed radically due to a new formal language that was used to represent mathematical propositions in an exact manner. For the first time, one seemed to be in the possession of a rigourous tool to express mathematical proofs completely and to distinguish valid proofs from invalid attempts of proofs. At last, the new techniques were developed up to a grade that allowed for new impossibility proofs that answer questions according to the limits of mechanical proofs in an exact mathematical manner that seemed to be open to philosophical speculation before.

However, the validity of this development in the foundations of mathematics rests on the presumption that mathematical propositions are adequately represented by the language of predicate logic. It is this presumption that Wittgenstein questions. Logic rests on the notion of propositional functions. Predicates are propositional functions and logical connectives are conceived as truth functions. A cornerstone of Frege's logical foundation of mathematics consists in his generalization of the mathematical concept of a function. To represent arithmetic within logic he had

This book elaborates a new picture of "old" pure predicate logic. Hardly any other theory seems to be as established as pure predicate logic. Thus, our enterprise needs some prior justification.

The starting point of my motivation for this project resulted from my investigation of Wittgenstein's early philosophy on the one hand, my teaching of classical logic on the other hand and coming to know the incompatibility of Wittgenstein's approach with the one of modern mathematical logic. Wittgenstein's understanding of logic in his earliest writings rests on the idea of a purely syntactic proof conception that significantly deviates from logical proofs in terms of derivations within a calculus. For him, the object of a logical proof is to represent the logical properties of predicate formulas, i.e. their truth conditions and their internal rela-

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[10] Cf. Kvasz (2008), p.67-84. Kvasz describes changes in mathematics by changes of the language of mathematics. As he mentions himself (p. 4), this idea is influenced by Wittgenstein. Kvasz applies this idea to the introduction of predicate calculus and set theory into mathematics in the second half of the 19th century. It is this step in the evolution of mathematics Wittgenstein rejects to.

tions, by the syntactic properties of an adequate notation. For this sake, such an adequate notation must be invented and a procedure is in question that converts predicate formula to the expressions of the ideal notation. This sort of a syntactic proof conception is motivated by what Wittgenstein calls his "main contention, to which the whole business of logical prop[ositions] is only a corollary", namely to distinguish what can be "expressed by prop[osition]s – i.e. by language – [. . .] and what can not be expressed by prop[osition]s, but only shown; which [. . .], is the cardinal problem of philosophy" (letter to Russell from 19.8.1919, CL 168, p. 124). In consequence, Wittgenstein maintains that it is not adequate to refer to logical properties by propositions, which are capable of being true or false corresponding to the facts. Instead, the only adequate way to express logical properties, and formal properties in general, is by representing them by syntactic properties of a symbolism itself. This approach is in conflict with metalogical proofs because those proofs refer to formal properties by propositions of a meta-language and represent those properties by propositional or characteristic functions.

However, Wittgenstein's "main contention" seems rather unconventional and his proof conception remains programmatic. While the exact understanding of his contention was open to a wide range of interpretation, the effort to elaborate his proof conception in logic was hardly ever taken. At last, Wittgenstein's ideas on logic, although of some philosophical influence, did not have a sustainable effect on the history of logic. In contrast, modern mathematical logic is well established and its methods and notions seem to be beyond doubt. Thus, the most decisive and well-informed assessment of Wittgenstein's approach claims that Wittgenstein's conception of logic is refuted by the evolution of modern logic as it seems to clash with the notions, methods and theorems of modern mathematical logic.

However, this judgement is unsatisfactory if one is interested in a profound evaluation of Wittgenstein's approach. No decisive mistake within Wittgenstein's conception is identified. Instead, the claim that it is refuted is rather based on the presumption of the superiority of the established paradigm and on the lack of a reasonable elaboration of Wittgenstein's alternative. Any substantial rejection of it should discuss the reasons of Wittgenstein's lifelong critique of metalogical proof methods and consider the question how to realize his proof conception in predicate logic, hereby taking into account Wittgenstein's own suggestions. Abstaining from doing so, is not only deficient from a systematic point of view but also from a historical and philosophical one. From a historical perspective, one wants to explain why mathematical logic has established itself. One fails to understand the lack of implicitness of the notions and methods of a well established discipline if one does not take into consideration rival approaches. From a philosophical perspective, Wittgenstein's proposal to make sense of what it means that "logical propositions are true in virtue of their form" was very influential and his philosophy of logic was taken as one of the most promising outlook at his time. However, his philosophy of logic cannot be understood nor evaluated without coming to know how his philosophy has to be implemented within logic.

Thus, we took on the task to elaborate systematically Wittgenstein's conception of pure predicate logic as far and as detailed as possible. We started with the intention to identify some concrete deficiency of his proof conception. However, we came up with the conclusion that, in fact, elaborating on his approach results in a paradigm of its own right. Thus, by elaborating in detail how predicate logic should look like from a Wittgensteinian point of view we intend to refute the common judgement that Wittgenstein's conception of logic is refuted by modern mathematical logic. This does not mean that we want to object to the traditional conception. Instead, we simply oppose it to an alternative. Hopefully, progress in coming to understand both sides will then make it possible to judge upon the merits and deficiencies of both paradigms.

2.2 Quantification and complex poles

Contrary to the method of truth tables, Wittgenstein also applies the ab-notation to quantified formulas. The ab-notation is significant because it is designed to apply to first-order logic and not merely to propositional logic. As Wittgenstein points out, the existential and the universal quantifier "occur simultaneously" in a proper notation of quantification that is based on the bipolarity of propositional functions (NL, p. 96). The simultaneous occurrence of quantifiers reveals their internal relation and makes it possible to identify internal relations between quantified formulas by syntactic features of their representations within the ab-notation. Wittgenstein provides two examples of his application of the ab-notation to quantified formulas (NL p. 96):

$$\forall x \phi x \quad a\text{-}\forall x\text{-}a\text{-}\phi x\text{-}b\text{-}\exists x\text{-}b \tag{2.1}$$

$$\exists x \phi x \quad a\text{-}\exists x\text{-}a\text{-}\phi x\text{-}b\text{-}\forall x\text{-}b \tag{2.2}$$

He deduces this notation from a formal explication of the truth conditions of the respective formulas, $I_F(A)$:

> The application of the ab-notation to apparant-variable propositions becomes clear if we consider that, for instance, the proposition "for all $x, \phi x$"

> is to be true when ϕx is true for all x's and false when ϕx is false for some x's.

Like in case of a-P-b and a-Fc-b this seems to be a rather trivial and non-illuminating explication. However, this is because of the primitiveness of the formulas. Furthermore, it is by far not trivial that Wittgenstein explains truth conditions of propositions that instantiate quantified formulas by making use of quantifiers. This significantly deviates from his account in TLP. In TLP he claims that the conditions of truth and falsehood of any proposition are reducible to the truth-possibilities of atomic propositions (TLP 4.4). Any proposition, he says in TLP 5, is a truth function of atomic propositions. Quantified propositions such as $\forall x\phi x$ or $\exists x\phi x$ are not regarded as truth functions of themselves. Quantified propositions are truth functions of a possibly infinite number of atomic propositions (TLP 5.52f.). In this respect, TLP corresponds to the interpretation$_{ME}$ of quantifiers in traditional logic. In TLP Wittgenstein believed erroneously that the formal explication of truth conditions of quantified propositions were reducible to truth functions of atomic propositions. That is the main reason why he abstained from his earlier ab-notation for first-order formulas in TLP. Yet, later he identified this as the "biggest mistake he had made in the *Tractatus*" (Wright (1982), p. 151); quantification cannot be explained by referring to "an infinite number" of atomic propositions (cf. p. *Lectures 1930-32*, p. 119, *Lectures 1930-33*, p. 298f., PG, p. 268-270). This account of quantification falls behind Wittgenstein's rejection of "the extensional infinite" in favor of a conception of the infinite on the basis of applying operations (cf. TLP 5.2523). More importantly, it makes it impossible to do justice to the claim that formal properties should be decidable by a proper notation. Wittgenstein's "biggest mistake" in the TLP was to abstain from his earlier ab-notation. The elaboration on the ab-notation in part II of this book intends to straighten out this mistake of the TLP.

Thus, we base the formal explication of truth conditions on a broader concept than the truth or falsity of atomic propositions. Within the ab-notation the truth of an atomic proposition P is represented by $a - P$, whereas its falsity is represented by $b - P$. These expressions consist of one pole assigned to a propositional variable. For simplicity sake, we write this pole always to the left if we only refer to single truth-possibilities. We call such expressions "complex poles" of propositional logic as they do not merely contain a- or b-poles. They correspond to literals of disjunctive normal forms in propositional logic. Whereas outmost a- and b-poles represent the possibilities of being true and false of propositions, complex poles define conditions of the truth and falsehood of propositions. Given the outmost a-pole is connected to the complex pole b-P this says that the possibility of being true of propositions instantiating a certain logical formula depends on P's falsehood.

However, complex poles are not confined to expressions of the sort a-P or b-P. Instead, in case of (2.1) the outmost a-pole is connected to $\forall x$-a-ϕx and the outmost b-pole is connected to $\exists x$-b-ϕx. As Wittgenstein's explanation manifests, these expressions play just the same role as complex poles of propositional logic: In case of $\forall x F x$ the possibility of being true depends on the condition identified by $\forall x$-a-ϕx, namely that "$\phi(x)$ is true for all x's", whereas the possibility of being false depends on the condition identified by $\exists x$-b-ϕx, namely that "$\phi(x)$ is false for some x's". The ab-notation is designed to produce formal explications of truth conditions of quantified propositions and it does so by use of quantifiers. The primitives of the ab-notation are not merely atomic propositions and their truth-possibilities but complex poles in general, which may contain propositional functions and quantifiers besides a- and b-poles.

It is all important to come to see that quantifiers do not vanish within the ab-notation. They are interpreted$_F$ as an irreducible part of the structure of quantified propositions. However, this structure cannot be identified by syntactic properties of common first-order formulas. The ab-notation intends to reveal the symbolizing properties of quantifiers within quantified propositions. This is done by introducing quantifiers "simultaneously" within the ab-notation and by identifying complex poles. The procedure of identifying complex poles within the ab-notation can be mapped by an equivalence procedure within the traditional syntax (cf. section 6.4.1). We will call the counter-parts of complex poles within the common syntax of first-order formulas "closed structures". Our decision procedure is definable as an equivalence procedure resulting in certain optimized distribute normal forms in terms of a disjunction of conjunctions of closed structures. This procedure assigns one and only one disjunction of conjunctions of closed structures to all equivalent logical formulas. Converting propositional formulas to reduced disjunctive normal forms, RDNF, according to the first step of the Quine-McCluskey algorithm realizes this idea within propositional logic. We generalize this kind of procedure by referring to closed structures as primitives of first-order logic.

The central idea of the ab-notation is to reduce the formal properties and internal relations of first-order logic to formal properties and internal relations of complex poles. These internal relations are more complicated than in case of propositional logic. Within propositional logic, the complex pole a-P is internally related to b-P and that is all: it is logically independent to all other complex poles of propositional logic. However, in case of (2.1) and (2.2) the four complex poles already stand in all kinds of internal, logical relations as the logical square shows (cf. figure 5.2). In order to avoid any ambiguities stemming from the use of variables we use numbers to represent the connection of the quantifiers to argument places of propositional functions within complex poles (cf. in detail p. **??**).

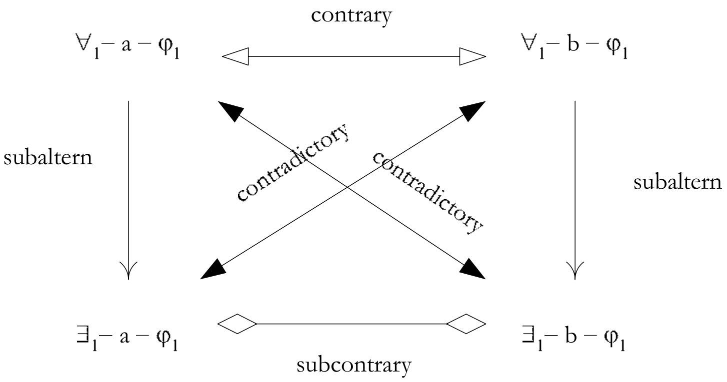

Figure 2.2: Logical square

As we will see, the internal relations of complex poles (or, likewise, of closed structures) are definable by systematic variation of their syntactic properties. Such a calculus determining internal relations of complex poles is reducible to applying thirteen derivation rules that do nothing but vary systematically syntactic features of complex poles such that the entirety of variations of symbolizing properties, which constitute relations of implication, is covered. The main contention of the ab-notation is that the internal relations of any number of complex poles is identifiable by a mechanical procedure within a finite number of steps on the basis of nothing but "computing syntactic properties" of complex poles.

2.3 Sentential connectives

Before we can go on to sketch how truth conditions of quantified expressions are identified by referring to complex poles, we must consider the interpretation of sentential connectives. Again, bipolarity causes an interpretation that radically differs from the traditional one. Ordinary syntax suggests to interpret$_{ME}$ connectives such as $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ as function symbols: $\neg P$, or $\neg(P)$, seems to be of the form $f(x)$; $P \wedge Q$, or $\wedge(P,Q)$, seems to be of the form $f(x,y)$. All that must be taken into account is that the argument-places are filled out by names of truth-values. Thus, sentential connectives are interpreted$_{ME}$ as truth-functions assigning a truth value in respect to the truth values of the connected propositions, cf. PM, p. 8:

> We may call a function $f(p)$ a "truth-function" when its argument p is a proposition, and the truth value of $f(p)$ depends only upon the truth-value of p.

Russell and Whitehead use p in this quotation as a variable for atomic as well as for molecular propositions. Thus, according to their understanding, the arguments in $\neg\neg P$ and $\neg P$ are different, whereas the function is the same, namely negation. P and $\neg\neg P$, however, are different truth functions with different arguments: P is a truth function of itself, while $\neg\neg P$ is a truth function, namely negation, of $\neg P$.

On the basis of Wittgenstein's conception of bipolarity, however, this interpretation$_{ME}$ of sentential connectives becomes obsolete. Wittgenstein interprets$_{F}$ connectives not symbols of as truth functions but of truth- or ab-operations. These operations assign a- and b-poles to a- and b-poles or to pairs of a- and b-poles. Thus, according to this understanding the proper representation of $\neg P$ is b-a-P-b-a and of $\neg\neg P$ it is a-b-a-P-b-a-b. Operations apply to "bases" (not to arguments). The bases of the operation of negation is the same in both of the cases $\neg P$ and $\neg\neg P$, namely an a- and a b-pole. Likewise, the bases of all dyadic ab-operation is always the same, namely aa, ab, ba, bb. The ab-operations differ only in assigning different poles to the same pairs of poles (cf. p. 122).

Wittgenstein sharply distinguishes between operations and functions. Contrary to functions, operations can be applied iteratively (TLP 5.251), and they can cancel one another (TLP 5.253). Operations do not refer to anything outside the symbols. Operations determine how to vary syntactic properties (TLP 5.241). They are syntactic instructions that must be carried out to determine formal properties or relations. Another symptom of the "basic mistake" of Old Logic is not distinguishing symbolically between operations and functions within the logical symbolism, cf. TLP 5.25 and WVC, p. 216f.:

> An empirical totality is traceable back to a *propositional function*; a system to an *operation*. The logical particles are truth-operations. Thus the meaning of the word 'or' is the operation that turns the sense of the propositions 'p', 'q' into the sense of the proposition 'p or q'. [...] An

> operation is completely different from a function. A function cannot be its own argument. An operation, on the other hand, can be applied to its own result. In mathematics we must always be dealing with systems, and not with totalities. Russell's basic mistake consists in not having recognized the essence of a *system* while representing empirical totalities and systems by means of the same symbol – a propositional function – without drawing any distinctions.

The all important difference to the traditional syntax and its I_{ME} is that on the basis of bipolarity and ab-operations it becomes possible to identify truth functions by syntactic properties within the ab-notation; one does not refer to truth values outside the symbols but to connections of poles within the expressions of the ab-notation. Thus, for example, it becomes possible to identify formally equivalent formulas, such as P and $\neg\neg P$, by identity of symbolizing properties within the ab-notation. Due to the understanding of sentential connectives as symbols of ab-operations ab-diagrams contain connections of poles instead of the signs $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$. ab-operations compute the connection of the outmost poles to the inmost poles. Within the ab-notation only the connection of the outmost to the inmost a- and b-pole symbolize. This connection represents the dependence of truth-possibilities to complex poles, which represent conditions of truth and falsehood. All intermediary poles do not contribute to the sense of the resulting ab-diagram, they only contribute to its construction. Thus, $\neg\neg P$ and P are represented by the same symbol within the ab-notation as what symbolizes in a-b-a-P-b-a-b and a-P-b is the same: in both cases the outmost a-pole is connected to the inmost a-pole of P and the outmost b-pole is connected to the inmost b-pole of P. This shows that the truth of propositions instantiating $\neg\neg P$ as well as of propositions instantiating P depends on nothing but on the truth of P, while their falsehood depends on nothing but on the falsehood of P. Likewise, the connection of the outmost poles to complex poles is the same in the ab-diagrams of equivalent such as formulas $\neg(\neg P \vee \neg Q)$ and $P \wedge Q$. Different sequences of sentential connectives within equivalent logical formulas result in identical connections of the outmost poles to inmost poles. Whereas the sequence of sentential connectives of the logical formulas does not provide a syntactic criterion to identify their equivalence, the ab-diagrams do provide such a criterion. This is due to I_F of sentential connectives as symbols of ab-operations, which makes it possible to identify identical symbolizing properties of ab-diagrams of equivalent formulas. We will call expressions of the ab-notation that only contain symbolizing properties "ab-symbols". In contrast, ab-diagrams are the immediate translations of logical formulas to the ab-notation that are not yet released from non-symbolizing properties such as intermediary poles.

The distinction of ab-operations and ab-functions is fundamental for New Logic and the correct understanding of the ab-notation. ab-operations (or truth-operations) are the formal interpretations, I_F, of sentential connectives, while ab-functions (or truth-functions) are the formal interpretations of logical formula, $I_F(A)$. ab-operations are carried out by constructing ab-diagrams; ab-functions are represented unambiguously by the resulting ab-symbols. ab-functions assign the outmost a- and b-pole to complex poles. They specify the truth conditions of propositions instantiating logical formulas. Any formally equivalent formulas represent the same ab- or truth-function. The identity of the ab-functions is identified by identity of the resulting ab-symbol, while the original equivalent logical formulas as well as their ab-diagrams might be different.

Nevertheless Old as well as New Logic interpret logical formulas as truth functions, this means something quite different in respect to the traditional syntax and its underlying material interpretation, I_{ME}, and in respect to the ab-notation and its underlying formal interpretation, I_F. Truth functions in terms of I_{ME} concern the dependence of the (actual) truth value of propositions instantiating a logical formula on the (actual) truth values of propositions instantiating parts of logical formulas. Thus, in case one traces back the dependence of the truth values of complex formulas to truth values of atomic propositions, propositions instantiating $\neg(\neg P \vee \neg Q)$ are true in case I_{MI} of P as well as of Q are true propositions such as, for example, $I_{MI}(P)$ = “Berne is the capital of Switzerland” and $I_{MI}(Q)$ = “Berlin is the capital of Germany”. In this case $I_{ME}(P) = T$ (i.e. $\Im(P) = T$) and $I_{ME} = T$ (i.e. $\Im(Q) = F$). Whereas propositions instantiating the above formula are false in case I_{MI} of P or I_{MI} of Q are false propositions, for example, $I_{MI}(P)$ = “London is the capital of Switzerland” or $I_{MI}(Q)$ = “Peking is the capital of Germany”. Thus, either $I_{ME}(P) = F$ (i.e. $\Im(P) = F$) or $I_{ME}(Q)$ (i.e. $\Im(Q) = F$).

Truth functions in terms of I_F, however, concern the dependence of the *possibilities* of being true or false of *one* proposition instantiating a logical formula on *unvaried* instantiations of complex poles. Thus, the values of the truth-functions are truth-possibilities. They are represented by the outmost a- and b-pole of an ab-diagram. The arguments of the truth-functions are complex poles, which specify conditions of truth and falsehood. In case of $\neg(\neg P \vee \neg Q)$ the outmost a-pole is connected to $a - P$ and $a - Q$, whereas the outmost b-pole is either connected to $b - P$ or to $b - Q$. Thus, due to I_{MI} of P as “Berne is the capital of Switzerland”, and I_{MI} of Q as “Berlin is the capital of Germany”, the proposition “Not

(Bern is not the capital of Switzerland or Berlin is not the capital of Germany)", which instantiates the formula $\neg(\neg P \vee \neg Q)$, is true if and only if Bern is the capital of Switzerland and Berlin is the capital of Germany, whereas it is false if it is false that Berne is the capital of Switzerland or if it is false that Berlin is the capital of Germany. One neither refers to different I_{MI} of P (or Q) in the explication of the conditions of truth and falsehood nor does one refer to I_{ME} of P (or Q). All one refers to are possibilities of being true and false (of propositions instantiating logical formulas) as they are specified by the syntactic structure of ab-symbols. To distinguish between the understanding of truth functions according to Old and New Logic, we label the first one "truth-value functions" and the latter "ab-functions". According to first understanding propositions instantiating logical formulas are names of truth values, which are assigned by I_{ME} to logical formulas. According to the second understanding propositions instantiating logical formulas are disguised "expressions of truth conditions" (TLP 4.431), which are revealed by I_F. According to I_{ME} primitive propositions are atomic propositions, which refer to a specific truth value that determine truth values of complex propositions. According to I_F primitive propositions are complex poles, which identify possible extensions that determine the possibility of being true or false of complex propositions.

Subsequent to his explanation of the ab-notation for $\forall x\phi x$ and $\exists x\phi x$ Wittgenstein notes "old definitions now become tautologous". Besides others, he refers to definitions *9.01 and *9.02 of PM:

$$\begin{aligned} &*9.01 \quad \neg\forall x\phi x = \exists x\neg\phi x \quad Df \\ &*9.02 \quad \neg\exists x\phi x = \forall x\neg\phi x \quad Df \end{aligned}$$

In generating the ab-symbols, the quantifiers of the logical formulas have to be assigned to the a-pole that is construed by carrying out ab-operations from the inside to the outside. The definitions become "tautologous" as the ab-symbols are identical in each case:

$$\begin{aligned} b\text{-}a\text{-}\forall x\text{-}a\text{-}\phi x\text{-}b\text{-}\exists x\text{-}b\text{-}a &= b\text{-}\forall x\text{-}b\text{-}a\text{-}\phi x\text{-}b\text{-}a\text{-}\exists x\text{-}a \\ b\text{-}a\text{-}\exists x\text{-}a\text{-}\phi x\text{-}b\text{-}\forall x\text{-}b\text{-}a &= b\text{-}\exists x\text{-}b\text{-}a\text{-}\phi x\text{-}b\text{-}a\text{-}\forall x\text{-}a \end{aligned}$$

Eliminating intermediary poles results in identical signs. Similar explanations hold for defining $\forall x\phi x$ by $\neg\exists x\neg\phi x$ and $\exists x\phi x$ by $\neg\forall x\neg\phi x$. In general, the ab-notation makes obsolete to introduce any quantifier or any sentential connectives as an indefinable. Wittgenstein rejects to "old indefinables" such as $\neg$, $\vee$, $\forall$ and $\exists$ because of their cross-definability (e.g. NL, p. 96, p. 101[3], TLP 5.42). Cross-definability shows their internal relations. This is revealed by the ab-notation. Within this notation, all what symbolizes are properties of structure. The meaning of the original signs is not reduced to a small number of them serving as indefinables. Instead, the meaning of the quantifiers and sentential connectives is reduced to contribute to the identification of truth-functions by syntactic properties of ab-symbols. According to I_F there is no need for interpretations going beyond the proper symbolism and refer to I_{MI} or I_{ME} of so called "logical constants". Thus, there are no indefinables or logical constants in terms of primitive signs with a certain meaning not expressible within the symbolism. Instead, there is nothing but formal interpretation of the symbolism expressed in its symbolizing properties. All that can count as indefinable within the ab-notation are the ab-symbols and the complex poles they contain (NL, p. 102[4]). These symbols are not primitive or indefinable in terms of having meaning not expressed by syntactic features but in terms of being the results of a purely syntactic procedure converting logical formulas in question to ideal symbols, which explain the formal properties of the initial formulas. The ab-symbols are a "neat translation" of a "cumbrous" explanation of truth conditions by ordinary language (cf. MN, p. 117[4]).

The process of converting logical formulas to ab-symbols starts with ab-diagrams. They make explicit the formal interpretation of the respective logical symbols. Logical formulas are converted to ab-diagrams by parsing the formulas form the inside to the outside according to their logical hierarchy. Wittgenstein provides several examples how to represent molecular formulas by ab-diagrams. In order to represent dyadic sentential connectives as ab-operations he uses brackets, which makes ab-diagrams rather clumsy. In a letter to Russell, he suggests to represent the formula $P \equiv P$ by the ab-diagram of figure 4.2.

This symbol differs from the ordinary formula $P \equiv P$ of the "old notation" by the fact that it allows one to identify its truth-possibilities by syntactic properties of the symbol. As Wittgenstein points out to Russell, tautologies of propositional logic are identified within the ab-notation by the syntactic property that the outmost b-pole is only connected to opposite inmost poles of the same atomic proposition (cf. CL, letter 32, p. 57). Thus, the above ab-diagram represents a tautology as the outmost b-pole is only connected to the complex poles a-P, b-P and b-P, a-P. b-P and a-P are contrary complex poles as one and the same proposition cannot be both, true and false. In order to identify the symbolizing properties of ab-diagrams of molecular formulas Wittgenstein refers to "classes of poles", cf. , NL, p. 102[4]):

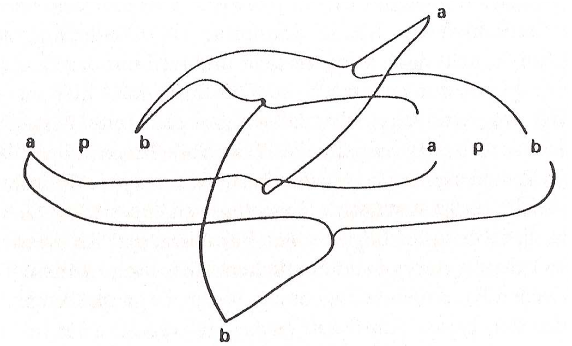

Figure 2.3: ab-diagram of $p \equiv p$ (CL, p. 57)

> Let n propositions be given. I then call a „class of poles" of these propositions every class of n members, of which each is a pole of one of the n propositions, so that one member corresponds to each proposition. I then correlate with each class of poles one of two poles (a and b).

In order to apply such a notation of classes of poles to first-order logic, I will generally refer to classes of complex poles, which I will label "pole-groups" in accordance to the terminology Wittgenstein uses in CL, letter 30, p. 53. The above ab-diagram, e.g., can be converted to the following a- and b-pole-groups:

$$a\text{-}\{a\text{-}P, a\text{-}P\},$$
$$a\text{-}\{b\text{-}P, b\text{-}P\},$$
$$b\text{-}\{b\text{-}P, a\text{-}P\},$$
$$b\text{-}\{a\text{-}P, b\text{-}P\}.$$

a- and b-pole-groups are the symbols that logical formula are converted to by the rules of the ab-notation. They serve as the syntactic identity criteria for determining truth conditions of propositions. The outmost a- and b-poles identify the possibilities of being true and false of a proposition whereas the complex poles of a pole-group identify conditions of their truth and falsehood. a- and b-pole-groups represent ab-functions: each a-pole-group assigns the a-pole to complex poles; each b-pole-group the b-pole to complex poles. The enumeration of a- and b-pole-groups form a finite list, which assigns the a- or the b-pole to finite numbers of complex poles. ab-symbols result from optimizing a- and b-pole-groups to the effect that all formulas of a class of equivalent formulas are converted to one and the same a- and b-pole-groups. This resulting pole-groups we call the "ab-symbol" of logical formulas. It is an ideal representation of the formal properties of the initial logical formula.

Propositions of the form $P \equiv P$ are tautologies as all the b-pole-groups contain contrary complex poles. Wittgenstein mentions to Russell that this criterion can be generalized to first-order logic (cf. CL, letter 30 p. 53 and letter 32, p. 57). He claimed that the ab-notation provides a decision procedure for first-order logic. As we will see in part II of this book, he is right. His criterion can be applied to first-order logic on the basis of a general notion of complex poles. Our decision procedure of first-order logic essentially consists in two procedures: (i) rules to convert formulas of first-order logic to a- and b-groups of complex poles, and (ii) operations to identify pole-groups containing contrary complex poles. (ii) is based on a general procedure to identify internal relations between pole-groups. It can be utilized to optimize a- and b-groups such that ab-symbols result as representatives of a class of equivalent formulas. As we will see this procedure is translatable into an equivalence procedure within first-order logic resulting in optimized distributive normal forms of closed structures.

Table 2.1 sums up the differences of I_{ME} and I_F of the logical symbolism. The difference cause that I_{ME} refers to some extension outside the symbols, whereas I_F is representable within the ab-notation.

	I_{ME}	I_F
prop. variable	truth value	primitive ab-function
predicate	set	primitive ab-function
name	object of an arbitrary domain	unspecified simple object
quantified expr.	truth-value function of an infinite number of atomic props.	ab-functions of complex poles containing quantifiers
connectives	truth-value functions	ab-operations
logical formula	truth value	ab-function

Table 2.1: Differences of I_{ME} and I_F

2.4 Converting logical formulas to ab-symbols

We will illustrate the application of the ab-notation to first-order logic in the following by referring to further definitions of PM that "become tautologous" within the ab-notation. By definition *9.03 - *9.08 of PM, disjunction is introduced in predicate logic:

$$\begin{array}{lrl}
*9.03 & \forall x\phi x \vee p = \forall x(\phi x \vee p) & Df \\
*9.04 & p \vee \forall x\phi x = \forall x(p \vee \phi x) & Df \\
*9.05 & \exists x\phi x \vee p = \exists x(\phi x \vee p) & Df \\
*9.06 & p \vee \exists x\phi x = \exists x(p \vee \phi x) & Df \\
*9.07 & \forall x\phi x \vee \exists y\psi y = \forall x\exists y(\phi x \vee \psi y) & Df \\
*9.08 & \exists y\psi y \vee \forall x\phi x = \forall x\exists y(\psi y \vee \phi x) & Df
\end{array}$$

We exemplify the application of the ab-notation by referring to *9.08. Figure 4.7 shows the ab-diagram of the definiendum $\exists y\psi y \vee \forall x\phi x$, figure 4.8 is the ab-diagram of the definiens $\forall x\exists y(\psi y \vee \phi x)$.

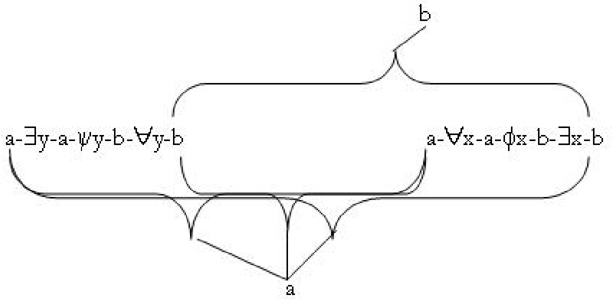

Figure 2.4: ab-diagram of $\exists y\psi y \vee \forall x\phi x$

To identify the respective complex poles in order to generate *pole-groups* it suffices in this case to connect the quantifiers only to those propositional functions containing the bound variable. In addition, intermediary poles are to be eliminated in order to yield the respective pole-groups. This already suffices to get identical result for both formulas:

$$a\text{-}\{\exists y\text{-}a\text{-}\psi y,\quad \forall x\text{-}a\text{-}\phi x\},$$
$$a\text{-}\{\exists y\text{-}a\text{-}\psi y,\quad \exists x\text{-}b\text{-}\phi x\},$$

$$a\text{-}\{\forall y\text{-}b\text{-}\psi y, \quad \forall x\text{-}a\text{-}\phi x\},$$
$$b\text{-}\{\forall y\text{-}b\text{-}\psi y, \quad \exists x\text{-}b\text{-}\phi x\}.$$

However, these pole-groups still contain non-symbolizing properties. The first two a-pole-groups are identical besides one pair of subcontrary poles. In this case, they can be replaced by one a-pole-group only containing the identical part. The same holds in case of the first and the third a-pole-group. In addition, we make use of a denoting argument positions of predicates by numbers and replace bound variables by the numbers of the argument places they refer to (cf. **??** for details. Thus, ambiguouties resulting from different variable notations are ruled out. From this, the following *ab-symbol* results:

$$a\text{-}\{\exists_1\text{-}a\text{-}\psi_1,\},$$
$$a\text{-}\{\forall_1\text{-}a\text{-}\phi_1\},$$
$$b\text{-}\{\forall_1\text{-}b\text{-}\psi_1, \qquad \exists_1\text{-}b\text{-}\phi_1\}.$$

This is the ab-symbol for all logical formulas equivalent with those of definition *9.08, e.g. also for $\exists x \forall y(\psi y \vee \phi x)$, which shows that the order of the quantifiers is not significant in this case. The resulting ab-symbol unambiguously identifies an ab-function. It is a neat expression of the truth conditions of propositions instantiating the initial formulas. It is the result of the formal interpretation, I_F, of the language of logic. By referring to instantiations of the formulas of 9.08* (and all their equivalents) by $I_{MI}(9.08*)$ one might paraphrase it roughly as follows: $I_{MI}(9.08)*$ are true if and only if $I_{MI}(\psi y)$ is true for some object at the first argument position or $I_{MI}(\phi x)$ is true for all objects at the first argument position and $I_{MI}(9.08)*$ are false if and only if $I_{MI}(\psi y)$ is false for all objects at the first argument position and $I_{MI}(\phi x)$ is false for some object at the first argument position."

Definitions *9.03 - *9.08 define disjunctions that do not occur within the scope of quantifiers by disjunctions that occur within the scope of quantifiers. Russell and Whitehead emphasize the priority of disjunctions occurring within the scope of quantifiers subsequent to their definitions, PM, p. 136:

> In virtue of these definitions, the true scope of an apparent variable is always the whole of the asserted proposition in which it occurs, even when, typographically, its scope appears to be only part of the asserted proposition. Thus when $(\exists x).\phi x$ or $(x).\phi x$ *appears* as *part* of an asserted proposition, it

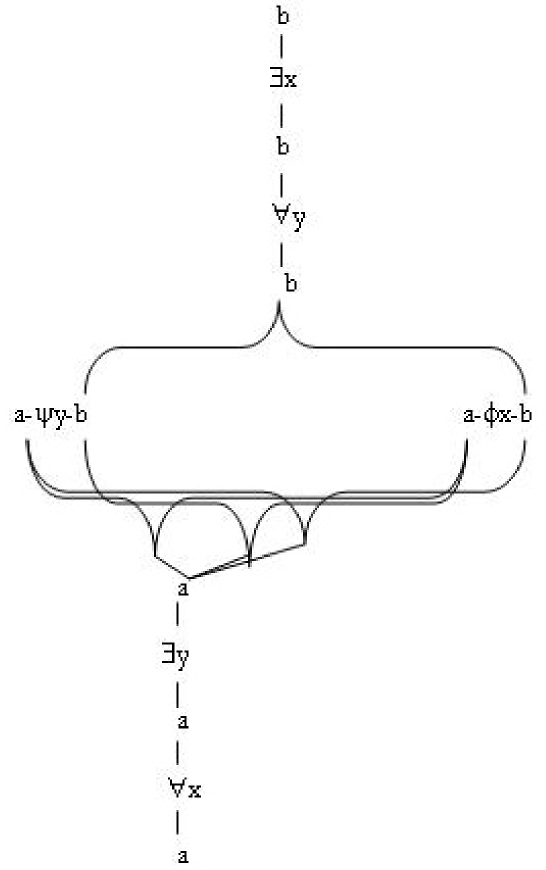

Figure 2.5: ab-diagram of $\forall x \exists y(\psi y \vee \phi x)$

> does not really occur, since the scope of the apparent variable really extends to the whole asserted proposition.

Thus, PM gives *priority to prenex normal forms*. This is significant for Old Logic. For example, the classification of so called “decidable” and “undecidable” classes of formulas refers to types of prenex normal forms (cf. Börger (1997)). However, prenex normal forms are incompatible with the $I_F(A)$ in terms of formal explications of truth condtions. Thus, contrary to the priority of prenex normal forms, the procedure of generating pole-groups imitates an equivalence procedure resulting in distribute normal forms of first-order logic. Contrary to prenex normal forms we minimize the scopes of quantifiers. The optimization of pole-groups, amplifies the first step of the Quine-McCluskey algorithm. Whereas Quine-McCluskey algorithm results in reduced disjunctive normal forms of propositional logic, our algorithm results in reduced disjunctive normal forms of first-order logic. This expansion becomes possible on the basis of the identification of internal relations between complex poles, or closed structures respectively. Thus, aiming for formal explications of truth conditions, New Logic gives priority to distributive normal forms.

2.5 Logical proof

The syntactic analysis of first-order logic accounts for a new understanding of logical proofs. According to New Logic, a logical proof is not a derivation of theorems from assumptions in order to prove the truth of theorems on the presumption of the truth of the assumptions. Instead, a logical proof is nothing but a syntactic analysis of logical formulas in terms of an equivalence procedure terminating in ideal symbols which identify the logical properties of the initial formulas by their syntactic properties.

In MN, p. 109[5] Wittgenstein describes the "procedure of the old Logic" as follows:

> This is the actual procedure of [the] old Logic: it gives so-called primitive propositions; so-called rules of deduction; and then says that what you get by applying the rules to the propositions is a logical proposition that you have proved.

Wittgenstein refers to logical proofs in terms of derivations within an axiomatic system. Mathematical logic is based on this proof conception. Frege's and Russell's systems satisfy this proof conception as well as modern sequence calculi or axiomatic systems extending first-order logic such as axiomatic systems of arithmetic: A formula is proven by deducing it from the axioms applying derivation rules. Wittgenstein does not deny that logically or mathematically true formulas can be identified by this procedure. However, he emphasizes that their truth cannot be proven this way. He goes on to say:

> The truth is, it tells you something about the kind of proposition you have got, viz that it can be derived from the first symbols by these rules of combination [. . .].

What is proven by the axiomatic proof procedure is simply the derivability of theorems from the axioms according to the rules of a calculus. The logical truth of the theorems, however, is not proven as it is based on the truth of the axioms and the correctness of the calculus. This is not denied within the framework of classical logic. Not Wittgenstein's comment that proofs within an axiomatic system are in need of a meta-logical justification is significant. Instead, it is the fact that his conception of New Logic opposes this common understanding of logical proofs that is remarkable. Throughout his life, Wittgenstein was opposed to understanding logical and mathematical proofs whose existence rested on axioms because one has to rely on some meta-logical, intuitive evidence if one wants to not only maintain the derivability of theorems, but their logical or mathematical correctness. In PG, p. 297 (cf. TLP 6.1271) he says:

> Logic and mathematics are not *based on* axioms, [...]. The idea that they are involves the error of treating the intuitiveness, the self-evidence, of the fundamental propositions as a criterion for correctness in logic.

Axiomatic proofs do not deliver a purely syntactic criterion for logical properties of arbitrary formulas of a formal system. The axioms are taken for granted without a formal proof. They hold an exceptional position within the system, but this position is not justified syntactically – the axioms are formulas within the system and do not differ essentially from other formulas. This can be seen from the fact that there are several correct and complete axiom systems for the same formal system. It can also be seen from the fact that not all axioms share some syntactic feature that identifies them as axioms. The common understanding of logical proofs in terms of derivation from basic assumptions depends on proofs of the logical truth of the axioms and of the correctness and completeness of a calculus relative to some previously defined semantics in terms of I_{ME}. Such proofs cannot be carried out within formal logic. Thus, the question arises regarding the meta-logical justification of an axiomatic calculus. Such a foundation necessarily exceeds the limitations of admissible evidence in logic. One objective of Wittgenstein's New Logic is to replace axiomatic proof procedures by a proof procedure that is not in need of such a meta-logical foundation. In TLP 6.1265f, he says:

> It is always possible to construct logic in such a way that every proposition is its own proof.
>
> All the propositions of logic are of equal status: it is not the case that some of them are essentially primitive propositions and others essentially derived propositions.
>
> Every tautology itself shows that it is a tautology.

That propositions of logic are "their own proof" or tautologies "show themselves" to be tautologies does not mean that there is no need for proofs in terms of syntactic manipulations of formulas in order to identify tautologies as tautologies. It only means that this can be done by relying solely on the formulas themselves as starting points of the proof instead of relying on axioms. In this respect Wittgenstein is looking for something similar to tableaux procedures such as Beth's or Smullyan's procedure (cf. Beth (1962), Smullyan (1995)). Contrary to these procedures, New Logic does not only aim for a procedure to identify tautologies, but for a procedure applying to "every proposition". In Wittgenstein's conception, proofs in terms of deriving theorems from assumptions are replaced by proofs in terms of converting formulas to symbols of an ideal notation. The purpose of the expressions of this ideal notation is not only to identify tautologies but to identify the truth conditions of propositions instantiating *any* formula by syntactic features. Again and again, Wittgenstein emphasizes that one has to identify tautologies "from the symbol alone" (TLP 6.113), or that one can "[recognize] in a suitable notation [. . .] the formal properties of propositions by mere inspection of the propositions themselves" (TLP 6.122). Axioms, i.e. formulas with an exceptional position within a logical system, are not needed in this conception because every formula "is its own proof," assuming its representation in a sufficient notation that identifies its truth conditions (TLP 6.1265, cf. 6.127f.). Put concisely, the proof conceptions can be compared as follows.

Proof conception of Old Logic:

$$\text{Axioms} \Rightarrow \text{theorems}$$

The formula in question marks the end of the proof. It has to be a theorem in order to be provable. Proofs of the truth conditions represented by formulas not being theorems are not available in this conception. This also applies to calculi of natural deduction that do not start their proofs by fixed axioms but prove argument schemata. Not to reveal the truth conditions expressed by logical formulas by syntactic features of ideal symbols is the objective of natural deduction proofs but to deduce a formula from certain assumptions. This conception is due to the objective to utilize logic in order to identify true propositions by derivation from propositions which are assumed to be true.

Proof conception of New Logic:

$$\text{Formula} \Rightarrow \text{ideal symbol}$$

The ideal symbol identifies the truth conditions expressed by the initial formula. In contrast to predicate formulas, its syntactic properties identify logical properties, i.e. the truth conditions of instantiations of logical formulas and their internal relations. The all important difference is that syntactic properties, i.e. properties recognizable in the symbols themselves, serve as identity criteria of logical properties. In this sense, a "syntactic foundation of logic" means to reduce predicate formulas to ideal symbols that allow one to identify the logical properties in question. It does not mean to define some axiomatic system that is correct and complete according to I_{ME}.

By exploring Wittgenstein's New Logic, it shall be demonstrated through purely logical means that understanding logical proofs in terms of derivations form assumptions is superfluous. It should be noted that it is not maintained that such proof systems are mistaken. However, in logic, their form is misleading in that it suggests that logic rests on some truth beyond symbols and their rule-governed manipulation. The form of such proof systems evokes problems at the very foundation of axioms or in the correctness and completeness of the proof system. According to Wittgenstein's point of view, these problems should be solved by changing the logical point of view rather than going beyond it. Thus, with the conception of New Logic, a certain philosophical point of view concerning the understanding and foundation of logic is at stake.

In contrast to calculi deriving theorems, the proof conception of New Logic rests on an equivalence procedure rather than derivations. The core question is to identify the truth conditions represented by the original logical formulas. Each proof step is part of the process to reveal those truth conditions by syntactic properties but it must not change the truth conditions expressed by the original formula. Internal relations between formulas such as formal implication then become transparent and identifiable by relations between the syntactic properties of their proper representations. In the framework of Wittgenstein's New Logic, laying down axiomatic calculi with certain meta-logical properties is not the first task of logic; the first step is solving the equivalence problem.

equivalence problem: The equivalence problem requires defining a general procedure such that the same symbol is assigned to every predicate formula of a class of equivalent predicate formulas, and different symbols are assigned to non-equivalent predicate formulas.

The decision problem is the special case of the equivalence problem concerning the identification of tautologies. Thus, its solution is implied in the solution of the equivalence problem.

The resulting symbols of a procedure solving the equivalence problem are "ideal symbols", in case of the ab-notation ab-symbols. Ideal symbols identify conditions of truth and falsehood expressed by predicate formulas as well as their internal relations unambiguously. By making use of Peirce's notion of "icon" and "iconic logic" (cf. Shin (2002)), one might label Wittgenstein's method "iconic proof conception" (as opposed to axiomatic proof conception), as it aims to replace expressions that do not represent truth conditions with expressions depicting truth conditions one-to-one by their syntactic features. However, it should be noted that Peirce distinguishes "symbols" that do not resemble what they represent from "icons". Wittgenstein's terminology differs significantly in this respect as he uses the term "symbol" and "symbolizing property" in order to identify those features of a sign that do resemble what is represented by that sign.

A logical proof involves the application a procedure solving the equivalence problem.

logical proof: A logical proof is a syntactic, rule-governed transformation from logical formulas to ideal symbols.

The purpose of such a proof conception is not to come to knowledge of some truth, but rather to reveal an objective, syntactic criterion in order to identify formal properties and relations of logical formulas. The complete realization of this proof-conception is the core problem of logic according to New Logic.

The proof conception of Old Logic is committed to the view that logic and mathematics aim for truth as any science does and that the truth of propositions is proven by derivation of axioms. New Logic, on the contrary, is not concerned with truth at all. From a logical point of view, all propositions are of "equal status"; contradictions do just as well as tautologies (TLP 6.1202). Logical proofs do not aim for truth but for revealing truth-possibilities according to New Logic. They do not serve to accumulate knowledge of truth values of propositions. Instead, they serve to identify their truth conditions. New Logic does not aim at increasing or justifying knowledge but at clarifying propositions. To achieve this, formulas of a deficient notation are to be replaced by mechanical means with an ideal symbol. This symbol identifies the truth conditions of the propositions by its syntactic properties and thus rules out the possibility of misinterpretations in terms of inadmissible instantiations of the initial logical formulas. A logical proof carries out a syntactic analysis in terms of $I_F(A)$ terminating in a formal explication of truth conditions of propositions. A proof in logic is an analysis of logical formulas just as a proof in mathematics is an analysis of mathematical propositions (cf. PB, p. 179). Analysis means converting expressions of deficient notations to ideal symbols. As I_F is prior to I_M, New Logic is based on nothing but the rules laid down to formally explicate truth conditions of propositions instantiating logical formulas. There is no reasonable distinction of syntax and semantic in this conception as every syntactic variation is motivated by a step in clarifying how the symbols contribute to the sense of propositions. The procedure identifies the symbolizing properties of logical formulas by first converting them to ab-diagrams and then striving off all insignificant syntactic properties in converting ab-diagrams to ab-symbols. Neither axioms nor any insights based on I_M need to serve as foundations within this conception. There is no foundations of logics or mathematics beyond syntactic analysis of the propositions or formulas in question. Using Wittgenstein's words, one can say "teach its rules and you have laid their foundation" (cf. PG, p. 297.).

2.6 Classification of propositions

Wittgenstein's ab-notation provides a criterion to distinguish meaningful propositions about facts from propositions that cannot said to be true or false according to a non-symbolic reality. Propositions are meaningful ("have sense") if and only if their truth conditions can be explained by non-contrary a- and b-pole-groups. Only propositions of this sort are "bipolar". "To have meaning (sense) *means* to be true or false" (MN, p. 113). In contrast, tautologies are logically true due to the absence of non-contrary b-pole-groups, whereas contradictions are logically false due to the absence of non-contrary a-pole-groups. Only bipolar propositions state something about reality; only in case of bipolar propositions reality decides upon their truth value. For this reason, Wittgenstein calls bipolar propositions and only bipolar propositions "real propositions". Only these propositions are "senseful" (in German "sinnvoll", which means "meaningful"). Other propositions only seem to state the existence of symbolized facts as long as one lays down as criterion calling them grammatically correct "true" or "false". However, in fact, they are true or false according to syntactic reasons; it is not reality but the proper analysis of the form of symbols that decides upon the truth value of those propositions. Wittgenstein calls such propositions "pseudo-propositions". His use of the terminus "pseudo-propositions" does not mean that those propositions are useless or somehow deficient. It only means that they must neither be analyzed nor formally represented as real propositions about objects although one might be inclined to do so because of their grammatical form and the univocal talk of truth.

Pseudo-propositions fall into two groups: Either they can be represented within the logical symbolism or not. Tautologies and contradictions comprise the first class of pseudo-propositions. They are logically true or false. Wittgenstein calls them “senseless” (“sinnlos” in German, which means “lacking sense”, cf. TLP 4.461[3]). In contrast, propositions attributing formal properties (or formal relations) fall into the second class. They are “nonsensical” (“unsinnig” in German, which means “meaningless”). Such propositions cannot be represented adequately within the ab-notation as only material properties can be represented by bipolar propositional functions. Thus, for example, (A) “It is raining or it is not raining” is a pseudo-proposition of the first sort as it is a tautology of the form $P \vee \neg P$ with the only contrary b-pole-group $\{a - P, b - P\}$. In contrast, (B) “The proposition ‘It is raining or it is not raining’ is a tautology” is a pseudo-proposition of the second sort as “_ is a tautology” is not a bipolar propositional function attributing a material property. The reason for this is that the proper syntactic analysis of “It is raining or it is not raining” already identifies this proposition as a tautology. Thus, (B) is true according to syntactic analysis and not according to some symbolized fact. It does not *state a property of objects* but *identifies a form*. That “It is raining or it is not raining” is a tautology cannot be stated by a real proposition. Instead, this is shown by expressing this tautology within the ab-notation. Within this notation, syntactic properties of the respective ab-symbol identify the formal property of being a tautology. Thus, (B) rather expresses a proper understanding of the syntax of (A) than a statement that could be denied.

This applies to all pseudo-propositions identifying formal properties or relations of propositions. In case of logic, these formal properties and relations are logical properties such as being a tautology / contradiction or logical implication / equivalence. Any $I_F(A)$ is also a pseudo-proposition as it expresses nothing but formal properties in terms of formal conditions of truth. This is exactly what is shown by converting logical formulas to ab-symbols. $I_F(A)$ gives proof of the usefulness of pseudo-propositions. They are illuminating as they clarify the syntax of propositions. They have explaining power in so far as they serve as explications of the truth conditions of propositions.

Pseudo-propositions involving logical properties might explicitly involve logical formulas, e.g. (C) “$P \vee \neg P$ is logically true”, or they might involve instantiations of formulas as in case of (A) and (B). Strictly speaking, pseudo-propositions involving instantiations imply both I_F as well as I_{MI}. As the paraphrase of the so called “logical constants” is fixed, it suffices to refer to $I_{MI_{cat}}$. This is done by a so called “realization” (“key” or “legend”, cf. Brun (2004, 2nd edition), p. 140). Tuples of a logical formula A + $I_{ME_{cat}}$ represent ordinary propositions. Pseudo-propositions on logical properties presume a decomposition in $< A$, $I_{MI_{cat}} >$ as they deal with the formal part of propositions. Thus, (B) is to read as "$< P \vee \neg P$, $I_{MI}(P) =$ It is raining $>$ is a tautology". According to I_F, $I_{MI_{cat}}$ must follow the principle of bipolarity and the principle of logical independence. To satisfy these principles, logical analysis of ordinary expressions is called for if ordinary propositions should be represented within the logical symbolism. However, the task of logic is to provide $I_F(A)$, whereas $I_F(A)$ + $I_{MI_{cat}}$ provide explanations of the truth conditions of propositions. The ab-symbol provides what Wittgenstein calls the "form of the sense of propositions" (TLP 3.13[4]). This ab-symbol and not the logical formulas, which are to be analyzed, reveals the real logical form of the proposition in questions.

According to the traditional view pseudo-propositions such as (C) are statements of meta-language. However, Wittgenstein's analysis of (C) as a pseudo-proposition significantly deviates from its understanding as a statement of meta-language. Meta-language statements do not differ syntactically from those of object-language. They only differ from statements of object-language by referring to expressions of the object-language as objects. They differ in the domain of objects; they do not differ in respect to syntax. The application of the logical syntax is not restricted to object-language within traditional logic. However, according to Wittgenstein's analysis the syntax of logic does not apply to pseudo-propositions of the second sort as they are not about objects at all but about forms. The subject-predicate form of pseudo-propositions of this sort does not determine their use as real propositions. One might object that logic can be applied to them as they can be called true or false. However, this is nothing but an articulation of the point of view of Old Logic that does not distinguish between formal and material truth. If one intends to explain the use of pseudo-propositions, one should instead take into account that the subject-term does not refer to objects but constitutes a form and that the predicate identifies certain features of this form. Pseudo-propositions call for defining the formal properties in question by decision procedures that identify those formal properties by syntactic properties of the subject term within a proper notation. Whenever such procedures are at hand, those propositions articulate a clear understanding (or misunderstanding) of syntax. Whenever they are not at hand, pseudo-propositions still call for a proper syntactic analysis. In consequence of such an analysis, pseudo-propositions become useful as they clarify syntactic problems and allow for new proof-strategies. In any case, pseudo-propositions cannot be represented adequately by logic as they do not state material properties of objects that alone are representable by bipolar propositional functions. The distinction of object- and meta-language does not avoid syntactic confusions. Moreover, this distinction itself suffers from the lack of distinguishing material and formal properties syntactically. Old Logic maintains that the distinction of meta- and object-language suffices to ensure proper formal representations of propositions. In contrast, according to New Logic the distinction between real and pseudo-propositions is the relevant distinction that avoids fallacies and misinterpretations stemming from syntactic confusions. It is the lack of this distinction that meta-mathematical proof methods share with paradoxes. Representing forms as objects and formal properties as material properties within a logical formalism is the cardinal source of fallacies prevailing Old Logic. No yet so rigorous defined syntax and semantics of a logical formalism can help it as long as it is just this syntax and semantics that is responsible for the fallacies. The use of logical formalization within modern logic does not avoid fallacies; it produces them.

Wittgenstein applies his distinction of real propositions and pseudo-propositions to mathematical propositions. He calls arithmetic equations "pseudo-propositions" (TLP 6.2). Equations can be solved by syntactic manipulations; they do not state anything about reality but identify formal relations. Wittgenstein denies any logical formalization of arithmetic that is cardinal for mathematical logic. He does not analyze arithmetics as a system of propositions representable within some arithmetical-logical axiomatic calculus. Instead, arithmetics consists in computation and this means operating with symbols. The paradigm of a good arithmetic proof is not a logical derivation from arithmetical-logical axioms. Instead, arithmetic proofs typically consist in equivalence transformation according to arithmetic operations in order to identify arithmetic properties by syntactic properties. According to Wittgenstein's analysis, numbers are not objects but forms. One does not *refer* to numbers by numerals. Instead, the proper notation of numerals identifies numbers as forms. The representation of numbers within the decimal system disguises their real form. The real form of natural numbers, for example, is revealed by generating them by successive application of the operation $+1$. Thus, for example, it is possible to identify the internal relation between 2, 3 and 4 by the syntactic relation of $1 + 1$, $1 + 1 + 1$ and $1 + 1 + 1 + 1$. Forms are generated and manipulated by operations. Numbers are not the arguments of functions. Addition, subtraction, multiplication or division are not functions assigning numbers to numbers. Instead, they are operations that determine the syntactic properties of numbers. Arithmetic properties are identified by syntactic properties of an adequate arithmetic notation. They are not material properties expressed by propositional functions. We will specify this conception of arithmetic in chapter **??**. For now, it suffices to recognize that New Logic rejects any logical formalization of arithmetic. Instead, it calls for identifying arithmetic properties as formal properties by syntactic features of proper arithmetic notations. Like logicism, Wittgenstein wants to distinguish arithmetic propositions from "synthetic propositions". However to characterize arithmetic propositions as "true in virtue of form", he draws a radical "antilogistic" consequence: Arithmetic is not based upon classical logic nor on some variation of classical logic, nor is it an expansion of logic. This is so because arithmetic is not based upon the concept of a function that applies to objects but on the concept of arithmetic operation that applies to numbers as forms. This consequence is not only in conflict with logicism but also with intuitionism (or constructivism) and formalism.

The denial of the concept of a function as basic for the foundations of mathematics is also incompatible with the method of modern metamathematics to represent properties of a formalism within an arithmetic-logical formalism. In particular, it rejects to define (i) enumerability by a function with the natural numbers as domain, (ii) computability by primitive recursive functions and (iii) decidability of (formal) properties by characteristic primitive recursive functions (Church's thesis). Instead, we will show in section **??** how to define these concepts on the basis of Wittgenstein's concept of operations. In fact, Wittgenstein envisaged an analysis of computation that differs significantly from that taken as basis in modern mathematical logic. Thus, we will define computation as what results from applying operations. Operations determine syntactic properties of symbols. Decidability of a formal property is therefore defined by the computability of a syntactic property that identifies the formal property in question. The key point is that only these definitions can do justice to the purely syntactic character of logic and mathematics. If one intends to give a clear account of what is meant by saying that a proposition is true "by virtue of form" one must give up the method of representing formal properties by propositional functions of form $\varphi(x)$ within the language of logic.

The discussed distinctions of propositions are recapitulated in figure 2.6.

According to the traditional point of view that goes back to Frege the language of logic is the proper language to speak about truth. Truth is primitive and there is no distinction between formal and material truth. Thus, logic is also the proper language to speak about formal truth. For this reason, mathematics as well as metalogical and metamathematical propositions are representable within logic. According to Wittgenstein, however, logic is merely a proper language to speak about symbolized facts. Logic is not the proper language to speak about formal properties. Instead, formal properties are to be represented by syntactic properties of a proper notation and not by propositional functions within the language of logic. For this reason, it is not adequate to represent mathematical or metalogical

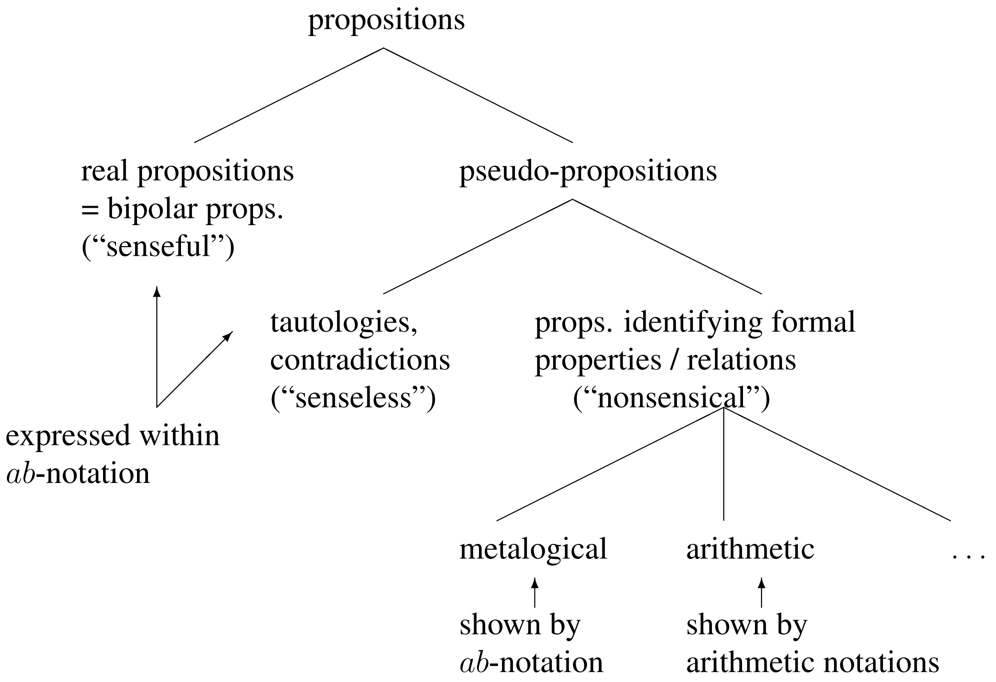

Figure 2.6: Classification of Propositions

and metamathematical propositions within the language of logic. Doing so, means to confuse propositions and pseudo-propositions. This, in turn, induces fallacies such as the proof of Church's theorem. This proof is not a fallacy in terms of a mistaken logical inference; it is a fallacy in terms of a syntactic confusion. The mistake of the proof cannot be identified within logic or by denying the "truth" of a certain assumption. Instead, the mistake lies in the presumed syntactic analysis of its assumptions. It is the application of logic to propositions that cannot be represented within the language of logic that is mistaken. Thus, in fact, so called "undecidability proofs" do not reduce the decidability of some formal property to absurdity. Instead, as we will explain in section **??**, correctly understood, they reduce the endeavor to absurdity to represent formal properties by propositional functions within the logical symbolism.

2.7 Summary

Traditional logic abstains from the sense of propositions. Only their reference to truth values is taken into account. In contrast, New Logic abstains from reference to truth values and considers only the truth conditions of propositions. The sense of propositions is identified with their truth conditions. Propositions are understood as expressions of truth-conditions (TLP 4.431). These truth-conditions are determined by the features of the logical formulas, which propositions instantiate. The task of logic is to formally explicate the truth conditions of propositions in respect to these features This is done by converting formulas of first-order logic to unambiguous a- and b-pole-groups, ab-symbols for short, by a purely mechanical procedure within a finite number of steps. The key point of New Logic is the idea to determine truth conditions of propositions by converting logical formulas to ideal symbols that identify unambiguously truth conditions by their syntactic features. Table 2.2 sums up the main differences of Old and New Logic.

	Old Logic	**New Logic**
interpretation	I_{ME}: refers to extensions (reference)	I_F: explicates truth conditions (sense)
principles	– bivalence – truth-value extensionality	– bipolarity – pole extensionality – logical independence
propositions	names of truth values	expressions of truth conditions
distinction of propositions	object- vs. meta-language	real vs. pseudo-propositions
logical proof	derivation from assumptions	equivalence transf. of logical formulas to ideal symbols
core problem	specifying correct and complete calculi	equivalence problem
normal forms	priority of prenex n. f.	priority of distrubitive n. f.

Table 2.2: Differences of Old and New Logic

Part II

Wittgenstein’s Programme

Chapter 3

Old vs. New Logic

This chapter describes the objective of Wittgenstein's New Logic in contrast to Old Logic. It will be shown that New Logic is not merely opposed to some peculiarities of Frege's or Russell's conceptions. Instead, it does significantly affect how predicate logic is commonly understood and taught on the basis of Frege's and Russell's works, especially in the context of modern mathematical logic. This concerns all main aspects of the common understanding of predicate logic in mathematical as well as in philosophical logic; namely its semantics, its syntax, its metalogical properties and its applicability to formalize ordinary or mathematical statements. As will be seen in this chapter, problems are relevant for Wittgenstein's conception of logic that are neither posed nor solved in the same manner within traditional logic. Three problems will be defined: (i) explaining the conditions of truth and falsehood of *wff* due to formal properties of their ideal representation, (ii) generating the totality of implications between *wff*s by operations applied to their ideal representation, and (iii) determining a general procedure that assigns the same symbol to all equivalent well formed formulae (*wff*s). As will be seen, the notions of formal properties and that of operations are fundamental for Wittgenstein's syntactical foundation of logic and mathematics. They stand in contrast to the use of "descriptions", which refer to properties not recognizable in the symbols themselves. Logical or arithmetical notions in terms of propositional or characteristic functions which *state* that certain formulae have some property without identifying this property by some syntactical property of the formulae themselves are descriptions. In this respect Wittgenstein's conception essentially differs from modern mathematical logic. This will be elaborated in section 3.1. The differences of Wittgenstein's approach of logic and the common one all stem from their differences in the conception of semantics, cf. section 6.3. These differences in the foundations of logic as well as in its semantic motivate a different conception of a logical proof that we contrast to the axiomatic proof conception of traditional logic in 3.2.

3.1 Extensional vs. intensional foundation of mathematics and logic

3.1.1 Operations, formal concepts, functions

In TLP, as well as in his writings between 1929 and 1934, Wittgenstein accuses Old Logic of confusing operations and functions (cf. TLP 5.25, cf. PT 5.005341, WVC, p. 213ff), as well as "formal concepts and concepts proper" (TLP 4.126[2]). Wittgenstein's objection does not primarily concern the lack of terminological specificity; instead it strikes at the fact that traditional logic does not do justice to these differences by syntactical differences of an adequate formalism. On the contrary, Old Logic represents them all by means of the same symbol, namely by propositional functions of form φx (TLP 4.126[4], 4.1272[8], 5.25, cf. WVC, p. 216). Instead of revealing formal differences by syntactical differences in an adequate logical formalism, traditional logic is guided by "the misleading present-day mode of expression" (WVC, p. 228).

This mode of expression articulates not only concepts proper, but also operations and formal concepts by means of ordinary statements that seem to be true or false independent of their syntax. Thus, for example, the statement "There are infinite even numbers" seems to be a true statement about the extension of the set satisfying the concept proper "x is an even number." However, according to Wittgenstein, the adequate expression of any infinitude is an operation that generates a series by iterative application. In the case of even numbers they are defined by the operation of adding 2 starting from 2: $[2,\xi,\xi + 2]$. The form of this expression differs significantly from the formal representation of concepts by means of predicate logic. For example, "There are at least two humans in this room" contains the concept proper "x is a human being in this room." In this case, it is adequate to use a propositional function, Fx, and formalize the proposition by $\exists x \exists y(Fx \wedge Fy \wedge \neg x = y)$ (or $\exists x \exists y(Fx \wedge Fy)$, as in the case of Wittgenstein's exclusive interpretation of predicate formulae).

Likewise, the statements "$P \vee \neg P$ is a tautology" ($\models P \vee \neg P$) or "2 + 2 is identical to 4" ($2+2 = 4$) seem to be true propositions about formal expressions satisfying the concepts "x is a tautology" and "x is identical to y". However, according to Wittgenstein's analysis, these expressions are not meaningful propositions, neither of object- nor of meta-language. They are "nonsensical pseudo-propositions" containing formal concepts. In his view, the adequate symbolic representation of the expressions in question makes it possible to identify their formal properties by their syntactical properties without any reference to non-symbolic features. In this sense, "is a tautology" and "is identical to" are redundant, superfluous supplements to $P \vee \neg P$ or to $2 + 2$ and 4: replacing the logical particles $\vee$ and $\neg$ by truth-operations as carried out in a truth-table and replacing 2 by $1 + 1$ and 4 by $1 + 1 + 1 + 1$ already shows the formal properties of the expressions in question. Likewise, "$P \wedge \neg Q$ is a tautology" or "2+2 equals 3" must not be analyzed as false propositions, but as applications of the concepts "... is a tautology" and "... equals ...", which are refuted by the syntax of their arguments. On the contrary, for example, "Fury is a horse" is a proper proposition that is true or false depending on the non-symbolic features because the syntax of the names that can be substituted for the variable in "x is a horse" is no sufficient criterion to identify whether the named objects are horses or not. Hence, "x is a horse" is a proper concept (= material concept), adequately symbolized by a propositional function that assigns the values TRUE or FALSE to its arguments according to non-symbolic features. According to Wittgenstein, only syntactical differences can do justice to the differences between operations, formal concepts, and material concepts, whereas Old Logic symbolizes them all by propositional functions.

Wittgenstein's criteria of distinguishing between formal and material concepts, and between operations and functions, are straightforward. A concept is formal iff its range is identifiable by syntactical means alone. That is, given an adequate notation, one can identify all "objects" satisfying the formal concept by the syntactical properties of their proper representation. In contrast, a concept is material iff this is not possible. That is, the concept is material iff one has to refer to non-symbolic features to identify the class of objects satisfying the material concept. Meta-logical concepts, such as "is derivable / provable", "is a tautology / contradiction", or "is consistent / satisfiable", that are usually conceived as propositional functions of meta-language are analyzed by Wittgenstein in terms of formal concepts that do not predicate anything about expressions. Instead, they are to be defined by syntactical properties of the adequate formal representation of the expressions possessing the formal properties in question. According to this conception, there is no meta-language in terms of statements about some object-language because all those pseudo-statements have to be traced back to purely syntactical features of a proper notation of object-language. Only analysis of object-language and its adequate formalization can answer what seems to be open questions of truth and falsehood according to the misleading surface grammar of meta-language. Any effective analysis makes statements in meta-language superfluous – only the explication of rules in order to manipulate the syntax of object-language is needed beyond object-language. The main task of logic and mathematics is to define the proper rules to solve logical and mathematical problems by means of a proper notation. If problems are not solved yet, this simply means that one has not defined the proper syntax for the expressions in question. We call this point of view, that the solution of the problems of logic and arithmetic depend on defining a proper notation, the "intensional point of view". As others, we emphasize by that label that Wittgenstein refers to operations / rules / laws instead of propositional functions / arbitrary functions / classes as constitutive for defining logical or arithmetic properties.[1] However, we also want to stress by that label that his position essentially rests on the invention of a proper notation to solve logical and arithmetic problems. In contrast, traditional logic holds an extensional point of view in assuming that there is logical and mathematical truth, independent of the particular mode of representation. It allows for "descriptions" in logic and arithmetic in the sense of phrases referring to non-symbolic properties to identify logical or arithmetic properties.

From the *Tractatus* on (cf. TLP 5.251), Wittgenstein repeatedly distinguishes functions and operations by the criterion of the possibility of iterative application, WVC, p. 217:

> An operation is completely different from a function. A function cannot be its own argument. An operation, on the other hand, can be applied to its own result.

Due to their possible iterative application, operations generate a series of internally related elements. These series are defined by an initial element, η, and an operation, $\Omega(\xi)$, that must be applied to generate an element from a previous one ξ. The form of such a definition is $[\eta, \xi, \Omega(\xi)]$. This defines a series, not in terms of an "infinite extension," but in terms of an unlimited application of an operation.[2] On the contrary, functions identify an "empirical totality", that is, an extension that is not given a priori by the definition of the function itself (WVC,

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[1] cf., e.g., Marion (1998b), p. 1-13, Rodych (2000), 283-285, Redecker (2006), p.149-155. Wittgenstein himself uses the "intensional / extensional" distinction to contrast his point of view with the traditional point of view, cf., e.g., PG part II, VII, 41 or RFM, part V, 34-40.

[2] Whenever we speak of "infinite series," we do not mean a series with an infinite extension, but a series generated by the unlimited application of an operation. The extension of any series is finite in reality, but the possibility to extend the series is infinite. Likewise, if we speak of an

p. 216). For functions, it is an open question whether certain objects satisfy the function or not, and by answering this question, the extension of the function is determined. *Empirical totalities* are sets of objects identified by functions, whereas operations construct *systems* of internally related elements. Numbers, as well as logical particles, are elements of systems, whereas trees, tables, and human beings are empirical totalities.

This kind of distinction departs from common terminology. Wittgenstein's terminus of a function corresponds to what is commonly called propositional functions or characteristic functions.[3] However, arithmetic functions, in terms of $f(x) = y$, are commonly understood as propositional functions, whereas, according to Wittgenstein's understanding, $f(x)$ in $f(x) = y$ usually represents an arithmetic operation (or a compound of several operations) and y the result of its application. Consequently, expressions in terms of $f(x) = y$ are not propositional functions (cf. TLP 6.2) and equations must be formalized within a logic (TLP 5.533) according to Wittgenstein. In this sense, Wittgenstein's critique does not only concern a special use of functions in the common sense, but the common notion of a function in general, since this can be traced back to propositional functions. "Number theoretical functions," such as $+$, $-$,$\times$ and $\div$, as well as "truth functions," such as $\neg$, $\wedge$,$\vee$, and $\rightarrow$, are operations according to Wittgenstein's terminology (cf. TLP 5.2341). Numbers and truth values, or, more precisely, ab-poles[4], are results as well as the bases for the respective operations. Arithmetic and logic are based on operations, not on functions (cf. WVC, p. 217). The notion of operations, not of functions is fundamental for Wittgenstein's syntactical foundation of logic and arithmetic.

Whereas functions determine the extension of a concept that is independent of its symbolic representation, operations vary the syntax of symbols. Both the base and the result of operations are symbols of a certain syntax. Operations have the form: "Generate expression B from expression A by carrying out the follow-

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"infinite number of objects" we do not mean that there is a number of objects greater than any finite number. Instead, we mean that, given any finite number of objects, a further object can be generated.

[3] According to common terminology, propositional functions are functions that assign truth values to their arguments. Characteristic functions are functions that assign 0 to numbers iff they satisfy some arithmetic property P and 1 iff they do not satisfy P.

[4] Wittgenstein rejects the presumption of "truth values" – that is why strictly speaking a- and b-poles serve as bases and results of truth-operations instead of TRUE and FALSE, cf. p. 132f. for a detailed analysis. "… is true" and "… is false", in turn, must be analyzed as operations, namely as the redundant operation of affirmation and the operation of negation. Hence, "p is true" means "p", while "p is false" means "$\neg p$" (cf. NB 6.10.1914, 27.11.1914).

ing syntactical transformation ...." Bases of operations serve as inputs, and the results of the application of operations are outputs that are computed from the inputs according to syntactical rules. An operation is defined by laying down how it converts the syntax of an input without defining a set of inputs.[5] Inputs are all and only those expressions to which the operation applies. If they do not have the syntactical feature that is varied by the respective operation they cannot serve as its input. In contrast, functions assign values to arguments of a certain domain without necessarily defining a rule that describes how the values are gained from the arguments. They state that their arguments have some property. In contrast, operations do not state anything; they only convert the *form* of the inputs.[6] They neither mark a certain form nor a content; instead, they specify a *difference* between forms (cf. TLP 5.24-5.254).

Operations can not only be applied iteratively, they also can be reversed such that the input is computed from the output, cf. TLP 5.253: "An operation can counteract the effect of another. Operations can cancel one another" e.g., subtraction counteracts addition, or negation cancels out negation. This, once more, shows that operations do not determine the content of expressions, but instead vary their form.

Operations specify some syntactic variation without referring to any other properties than syntactic properties of the expressions in questions. By "syntactic properties" of an expressions we refer to the type of signs and their relation in this expression. We do not refer to internal relations to *other* expressions such as the derivability of one formula from another. Operations define variations in terms of adding or eliminating some syntactical feature or replacing one syntactical feature with some other syntactical feature. However, these variations must not depend on some non-syntactical property outside the symbols in question. Thus, in case of logic any derivation rule referring to some other derivation in its definiens, is no logical rule in terms of an operation. The following well known derivation rules, for example, are no logical operations, because they all depend on the existence of certain derivations:

RAA**:** $\Gamma, (A, \Gamma \vdash B \wedge \neg B) \vdash \neg A.$

$\vee E$**:** $A \vee B, \Gamma, \Delta, (A, \Gamma \vdash C), (B, \Delta \vdash C) \vdash C.$

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[5] cf. Marion (1998b), p. 12 and 23 who cites NB, p. 90, 22.11.16: "The concept of the operation is quite generally that according to which signs can be constructed according to a rule."

[6] NB, p. 81, 17.8.1916: "An operation is the transition from one term to the next one in a series of forms."

$\rightarrow I$**:** $\Gamma, (A, \Gamma \vdash B) \vdash B$.

$\forall I$**:** $A(t), (\Gamma \vdash A(t)) \vdash \forall v A(v)$ if Γ does not contain t.

$\exists E$**:** $\exists v A(v), \Gamma, (A(t), \Gamma \vdash C) \vdash C$ if $\exists v A(v)$, C and Γ do not contain t.

Any reference to assumption lists as it is typical for natural deduction calculi is also excluded by the claim that derivation rules in terms of operations must not refer to anything else than syntactic variations of the formulae themselves. However, logical derivation rules specifying syntactical transformations *without referring to anything else than the syntactic features of the formulae* are operations. Thus, for example, the following well known derivation rules are operations:

DN**:** $A \dashv\vdash \neg\neg A$.

$\vee E$**:** $A \vdash A \vee B$.

$\forall E$**:** $\forall v A(v) \vdash A(t)$.

$\exists I$**:** $A(t) \vdash \exists v A(v)$.

To derivation rules of this kind Wittgenstein refers in TLP 6.126 when he speaks of logical operations as “rules that deal with signs” that are successively applied to prove logical propositions. New Logic only allows for calculi exclusively making use of derivation rules in terms of operations. Our elaboration of New Logic in part two of this book will be based upon such a calculus specifying all possible minimal syntactic variations between a certain subclass of predicate formulae to which all predicate formulae are reducible by equivalence transformation.

In case of arithmetic, any definition of a series of numbers by some property for which it must be decided first, whether some number has it or not, does not satisfy Wittgenstein’s criterion to define series of numbers by operations. Thus, for example, the series of primes is not yet well-defined by saying that a number is a member of the series if it is only dividable by itself and by 1. According to Wittgenstein, the usual definition of primes does only define what *a* prime is, but it does not define *the* infinite series of primes (cf. Lampert (2008) and section 3.1.4). Likewise, defining the series of even numbers as those numbers that can be divided by 2 is not yet a proper definition of a series of numbers. Instead, the members of a series of numbers must be defined by purely arithmetic formulae allowing for calculating the nth member from previous members or from n. In case of the series of even numbers, for example, $a_{n+1} = a_n + 2$ with $a_1 = 2$

satisfies the former and $a_n = 2 \cdot n$ the latter criterion. In the former case we yield the series $2, 2+2, 2+2+2, \ldots$. In the latter case, the operation defining the members of the series, is simply adding 1. Thus, we generate the series $2 \cdot 1$, $2 \cdot (1+1)$, $2 \cdot (1+1+1)$ .... Replacing the members of both series by the resulting decimal numbers then yields 2,4,6,8,... in both cases. By definitions of this kind, one does not have to consider each natural number and ask whether it has some property or not. Instead, one is able to define them by systematic syntactic variations of the expressions of the series itself. Thus, for example, the following definitions of the respective series satisfy Wittgenstein's criteria:

$2^1, 2^{1+1}, 2^{1+1+1}, \ldots$	$a_n = a_{n-1}^{+1}$ or alternatively, $a_n = 2^n$.
0,1,0,1,0,1 ...	$a_1 = 0$, $a_2 = 1$, $a_n = a_{n-2}$ or, alternatively, $a_n = \frac{1}{2}[1 + (-1)^n]$.
0, 1, 1, 2, 3, 5, 8, 13, 21, ...	$a_1 = 0, a_2 = 1\ a_n = a_{n-2} + a_{n-1}$ or, alternatively, $a_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]$.
$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \ldots$	$\frac{a_1}{b_1} = \frac{1}{1}, \frac{a_n}{b_n} = \frac{b_{n-1}}{a_{n-1}+b_{n-1}}$.[7]

If we would have written "2,4,8, ..." instead of "$2^1, 2^{1+1}, 2^{1+1+1}, \ldots$" in the first line, we were not able to define this series by an operation in terms of defining following members by a pure syntactic variation of previous ones. Instead, one must calculate the respective decimal number from some expression 2^n in addition. This shows that defining operations essentially depends on suitable modes of representation. This also holds for the following series. Strictly speaking, one must not represent the decimal numbers of the respective arithmetic expressions. Instead, one must represent the arithmetic expressions generated by applying the operations without calculating the resulting decimal number to show that the series is definable by nothing but a systematic syntactic variation. For the sake of simplicity and transparency and for the sake of providing alternative definitions for the same series, we abstain from doing this here.

In contrast to the above definitions, definitions of series of numbers by characteristic functions are examples of descriptions. The following definitions, for example, are all rejected by Wittgenstein (PG, Part II, §42):

F**:** the binary fraction $0.a_1a_2a_3\ldots$ with $a_n = 1$ if $x^n + y^n = z^n$ is solvable for n, otherwise $a^n = 0$.

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[7] There is no possibility to define this series from n, cf. Waismann (1936), p. 103.

P**:** the binary fraction $0.a_1a_2a_3\ldots$ with $a_n = 1$ if n is prime, otherwise $a_n = 0$.

$\pi^{777\to000}$**:** the decimal number $3.a_1a_2\ldots$ with $a_na_{n+1}a_{n+2} = 000$ if $a_na_{n+1}a_{n+2}$ of $\pi = 777$, otherwise $a_n = a_n$ of π.

C**:** the binary fraction $0.a_1a_2a_3\ldots$ with $a_n = 1$ if a coin shows head, otherwise (= if it shows tails) $a_n = 0$.

According to our interpretation, the reason for rejecting all these definitions is simply that they attempt to define a number series by a characteristic function. This is incompatible with Wittgenstein's claim to define number series by operations in terms of arithmetic formulae allowing to generate following members from previous ones or from n. We will explain Wittgenstein's rejection of so called "pseudo-irrationals" in more detail in section 3.1.4 in connection with his view on real numbers in general and his rejection of the number P in particular.

Wittgenstein calls any definitions in arithmetic or logic that refer to properties other than syntactic properties of logical or arithmetic expressions "descriptions". His way of a syntactical foundation of logic and arithmetic does not allow for descriptions. The prevalent use of descriptions in mathematical logic is a symptom of its deficiency according to New Logic. The task for a future logic and arithmetic satisfying Wittgenstein's standards is to reduce descriptions to operations whenever possible.

Wittgenstein's distinctions between operations and functions, as well as between formal and material concepts, are at the heart of his purely syntactical foundation of logic and mathematics. These distinctions are not merely tools of some philosophical attempt to analyze the specific character of logic and mathematics. On the contrary, its consequences are radical and stand in sharp contrast to modern mathematical logic. This is often overlooked in the literature. Many do not recognize the incompatibility of Wittgenstein's mathematics and logic with modern mathematical logic because of two reasons: (i) the erroneous assumption that Wittgenstein's notion of operation is equivalent to the modern notion of primitive recursive functions[8], and (ii) the lack of recognition that Wittgenstein's criticism does not primarily consist of a terminological specification, but instead amounts to a rejection of the formalism of modern mathematical logic. We will illustrate this by contrasting his concept of operations and formal concepts with the notion of primitive recursive functions (p.r. functions) in general first (sections 3.1.2 and

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[8] cf. Marion (1998b), p. 99 and pp. 105-109, Redecker (2006), p. 189f., cf. Sundholm (1992), p. 65, cf. also Landini (2007), p. 128 who identifies Wittgenstein's notion of operations with the mathematical notion of a function in general.

3.1.3). Then we will cement this by Wittgenstein's rejection of a special primitive recursive function, P, in section 3.1.4. Finally, we will draw the consequences for the elaboration of New Logic (cf. sections 3.1.5 to 3.1.7).

3.1.2 Operations vs. p.r. functions

P.r. function are defined as follows[9]:

1. The successor function, S, the zero function, $Z(x) = 0$, and the identity function, $I_i^k(x_1, \ldots x_k) = x_i$, for each k and for each i, $1 \leq i \leq k$ are p.r.,
2. if f can be defined from the p.r. functions g and h by composition, substituting g into h, then f is p.r.,
3. if f can be defined from the p.r. functions g and h by recursion, then f is p.r.,
4. if g is a regular p.r. function, and f can be defined from g by regular minimization, then f is p.r.[10],
5. nothing else is a p.r. function.

The successor "function" is equivalent to an operation – the series of natural numbers can be defined by its iteration: $[0, \xi, S(\xi)]$. However, that primitive recursiveness and operations are not equivalent, becomes obvious with the other two initial functions: $Z(x) = 0$ and, as the most simple example of the identity function, $I(x) = x$. These two functions are p.r. by definition, but they are not operations because they do not imply any iteration or recursion. This can be seen from the fact that, in both cases, the values of the functions do not define further arguments. Furthermore, as will be explained in more detail below (cf. p. 88), neither functions refer to expressions, nor do they provide any syntactical routine for manipulating expressions, as is essential for operations (cf. p. 78). They simply define two functions, without any advice on how to generate the values given the arguments. They do not imply certain calculation types, such as $x \times 0 = 0$ or

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[9] cf. Smith (2007), p. 267. Contrary to Smith, and like many other authors, we simply use "recursive" instead of "μ-recursive".

[10] By regular minimization functions are considered using a "do until" search procedure that calculates the number $f(x)$ which is the least y such that $g(\vec{x}, y) = 0$. $\vec{x}$ stands for n variables. g is the characteristic function of the property G. By referring to g as a regular p.r. function, it is ensured that there is, for all values of $\vec{x}$, a least y such that $g(\vec{x}, y) = 0$.

$x + 0 = x$ to compute their arguments. Instead, these expressions are merely two inessential “modes of representations” of the functions $Z(x) = 0$ and $I(x) = x$. These functions are meant to be propositional functions that are true for all numbers independent of some special instruction to compute the values given their arguments. They are defined extensionally. That the definition of a p.r. function is based on these propositional functions already suggests that it is part of a conception that is incompatible with that of Wittgenstein.

The idea of defining a function by regular minimization is also not part of an operation, as a “do until” procedure implies that one decides, one by one for natural numbers, whether they satisfy some property in order to identify the least y satisfying some property. Such a procedure implies that one first has to refer to some extra-symbolic property to proceed while the application of operations solely depends on the syntax of their bases. Contrary to p.r. functions, operations generate a series of elements without deciding whether some objects satisfy some property. Operations are not based upon characteristic functions (cf. p. 79 and section 3.1.4).

The striking similarity between p.r. functions and operations seems to relate to recursion. Furthermore, operations might also be composed such that the result of a certain operation might itself serve as the base of some other operation. Thus, there seems to be a similarity between the chain of definitions by recursion and composition in p.r. functions, and a series of applications of operations. However, this similarity is not an equivalence because “recursion” does not mean the same thing in both cases. For operations, recursion means that the result (output) of an application of an operation can itself be the base (input) of the operation. Thus, a series of results of applications of operations is generated where the $n+1$-th result is the successor of the n-th result. If recursion meant the same for p.r. functions, the values of those functions would again be their arguments. Yet, this is not the meaning of recursion in p.r. functions. In p.r. functions, recursion means that the value of the argument of the primitive function for $S(n)$ is defined by referring to the value of the very same function for n. Hence, the recursion refers to the succession of natural numbers as given by the successor function, and not to a succession of the application of the function itself. This also becomes clear from the fact that, in case of composition, it is referred to the value of the very same function for n *as argument of some other function*. We will illustrate these differences by referring to the definition of addition as a p.r. function as follows:

$$x + 0 = x$$
$$x + S(y) = S(x + y)$$

Here, the identity function is the initial function. The second clause defines addition by recursion and composition. Thus, the value of addition for the argument $S(y)$ is defined by referring to the value of addition for y as argument of some other p.r. function, namely S. Thus, addition is defined by the successor function and the identity function. These two clauses define addition *implicitly*. They serve as axioms in an axiomatic system of arithmetic, those axioms containing addition. If we do not abstain from quantification, we obtain Axiom 3 and 4 of Peano Arithmetic PA from the two definitions:

$$\forall x(x+0=x)$$
$$\forall x\forall y(x+S(y)=S(x+y)).$$

Admitting quantification again demonstrates that p.r. functions are used as propositional functions. In contrast, Wittgenstein rules out any use of logical quantification by formalizing arithmetic propositions, cf. p. 89 below. This, again, is clear evidence that his conception denies the conception of p.r. functions.

In the framework of PA, equations containing addition can be proven by derivation from these axioms. The proof, e.g., of $2+2=4$, i.e. $SS0+SS0=SSSS0$ in the notation of PA, is as follows:

no.	**formula**	**rules**
(1)	$\forall x(x+0=x)$	Ax. 3
(2)	$\forall x\forall y(x+S(y)=S(x+y))$	Ax. 4
(3)	$SS0+0=SS0$	1 $\forall E$
(4)	$\forall y(SS0+S(y)=S(SS0+y))$	2 $\forall E$
(5)	$SS0+S0=S(SS0+0)$	4 $\forall E$
(6)	$SS0+S0=SSS0$	3,5 =E
(7)	$SS0+SS0=S(SS0+S0)$	4 $\forall E$
(8)	$SS0+SS0=SSSS0$	6,7 =E

The proof starts with axioms and ends up with the theorem to be proven. Wittgenstein does, of course, not negate that PA identifies correct equations. However, according to Wittgenstein, axiomatic proofs rest on inadequate formalizations, which are incompatible with his intention to refer to nothing but syntactical features to identify the formal properties in question. The definition of addition as a p.r. function is a symptom of traditional logic's misconception of arithmetic.

By the scanty remarks on mathematics in the TLP, it already becomes clear that Wittgenstein opposes his iconic proof conception, not only in logic, but also in arithmetic, to an axiomatic proof conception.[11] While his interpretation of logical particles, e.g. $\neg$, $\wedge$, $\vee$ and $\rightarrow$, as operations is at the heart of his proof conception in logic, the interpretation of $+$, $-$,$\times$ and $\div$ as operations is at the heart of his proof conception in arithmetic. In both cases, the all-important point is that his analysis calls for different symbolic representations that eliminate the symbols ($\neg$, $\wedge$, $\vee$, and $\rightarrow$, as well as $+$, $-$,$\times$, and $\div$) in order to carry out proofs in terms of converting formulae of a certain deficient formalism to a proper notation. As our main interest is to elaborate Wittgenstein's proof conception in logic, we will demonstrate this in short only for addition. As our intention is merely to contrast Wittgenstein's approach to arithmetic with that of PA, we keep things as simple as possible and, as such, do not refer to any other arithmetic operation than addition.

Wittgenstein's conception of arithmetic is based on his definition of numbers as exponents of operations. This can be understood as a generalization of defining numbers by the successor "function", which is merely a special operation according to Wittgenstein's point of view. Thus, as PA symbolizes numbers through use of the successor function, Wittgenstein represents them with a certain number of applications of operations in general. From this, we can define 2 by applying some operation twice and 4 by applying some operation four times:

$$\Omega^{2}\text{'}\eta = \Omega\text{'}\Omega\text{'}\eta \qquad \text{Def. 2}$$
$$\Omega^{4}\text{'}\eta = \Omega\text{'}\Omega\text{'}\Omega\text{'}\Omega\text{'}\eta \qquad \text{Def. 4}$$

For simplicity, we deviate from Wittgenstein's derivation of these two equations, from the definitions $0 + 1 + 1 = 2$, $0 + 1 + 1 + 1 + 1 = 4$, and from the inductive definition $x = \Omega^{0}\text{'}x$ and $\Omega\text{'}\Omega^{v}\text{'}x = \Omega^{v+1}\text{'}x$ in TLP 6.02. Contrary to the symbols on the right hand side of the above definitions, the symbols on the left hand side are part of a deficient syntax that lacks the proper multiplicity to identify the internal relations between numbers by the syntax of their representation.

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[11] cf. TLP 6.02-6.031, 6.2-6.241. Our reconstruction of Wittgenstein's proof of $2 + 2 = 4$ is based on his definition of numbers in 6.02, his proof of $2 \times 2 = 4$ in 6.241, as well as his proof sketch of $2+2 = 4$ in the earlier drafts of the TLP, TS 204 and TS 203, cf. Graßhoff (2004), p. 169 and p. 309. However, we deviate from the proof sketch given in TS 203 and TS 204 by making use of the Ω-notation and a definition of addition similar to the definition of multiplication in 6.241. Contrary to Frascolla (2000), who provides a thorough formal analysis of arithmetic proofs according to the TLP, we assume that Wittgenstein aimed for an iconic and not for an axiomatic proof conception.

As addition is an operation, it must be defined by some variation in the syntax of formal expressions. Just as in the case of numbers, addition is defined by referring to its syntactical representation in a proper notation.

$$\Omega^{m+n}\text{'}\eta = \Omega^{m}\text{'}\Omega^{n}\text{'}\eta \quad \text{Def. +}$$

Here, addition is not defined implicitly by axioms, but rather by an explicit definition, with which we can eliminate the sign of addition. Thus, addition is explained as the application of a certain iteration of operations to a certain iteration of operations.

The above three definitions constitute proofs of equations containing numbers and addition in terms of converting expressions of a deficient syntax to expressions of an ideal syntax.

Equations like $2 + 2 = 4$ are part of a formalism that cannot decide by the syntax of the expressions to the left and to the right of the identity sign whether the formal concept of arithmetic identity applies to its arguments. As the expression on the left hand side of the equation represents a number just like the right hand side, and as numbers are analysed as exponents of operations, one must replace $2 + 2 = 4$ by $\Omega^{2+2}\text{'}\eta = \Omega^{4}\text{'}\eta$ in order to apply Wittgenstein's definitions. This is similar to replacing the equation $2 + 2 = 4$ by $SS0 + SS0 = SSSS0$ in PA.

As in Wittgenstein's proof of $2 \times 2 = 4$, we prove $2+2 = 4$ by first converting the expression on the left hand side of the equation to its ideal symbolization, and then converting this ideal symbolization to the expression on the right hand side. We marked this strategy by noting "l.h.s" in the first and "r.h.s" in the last line of the proof. Thus, the proof does not start with axioms of a proof system, but with the left hand side of the equation, and it does not end with the theorem to be proven, but with the right hand side of the equation. Hence, the syntactic criterion of the proof is not the derivation of the equation in an axiomatic system, but rather it is the possibility of converting the expression on the left and on the right hand side of the equation to the same ideal expression, namely

$$\Omega\text{'}\Omega\text{'}\Omega\text{'}\Omega\text{'}\eta$$

, by using definitions that reveal the proper syntax of arithmetic expressions. The identity sign is not part of expressions used in the proof any more; identity is not expressed by a statement of object language but by identity of ideal expressions. By applying the given definitions we yield the following proof of $2 + 2 = 4$, i.e. $\Omega^{2+2}\text{'}\eta = \Omega^{4}\text{'}\eta$:

no.	**formula**	**rules**
(1)	Ω^{2+2}'η	l.h.s.
(2)	Ω^{2}'Ω^{2}'η	Def. +
(3)	Ω'Ω'Ω^{2}'η	Def. 2
(4)	Ω'Ω'Ω'Ω'η	Def. 2
(5)	Ω^{4}'η	Def. 4, r.h.s

As any step of the proof does nothing but convert the formula of the preceding line, one need not refer to previous lines. Consequently, numbering the lines is not necessary. By applying Def. 2 to Ω^{2}'Ω^{2}'η in line (2) η in Ω^{2}'$\eta = \Omega$'Ω'η (= Def. 2) is replaced by Ω^{2}'η.[12] Contrary to Wittgenstein's proof in 6.241, we do not make use of parentheses. Parentheses can be eliminated as they are no symbolizing property in the Ω-notation, cf. TLP 6.231f. and section 4.2 for the notion of 'symbolizing property'.[13] Accordingly, eliminating parentheses need not be justified by a further definition in the Ω-notation. Parentheses may be helpful to understand the proof, just as it might be helpful to signify which expression is replaced for η in Def. 2. However, this is not an essential part of the proof, but only of the explanation (the "prosa") of the proof. Likewise, we tacitly eliminated parentheses in the PA-proof.

The difference between this proof and that in PA is not that they give different results, of course. Rather, the difference is in the underlying conception of arithmetic expressed in different symbolizations and different proofs.

3.1.3 Formal concepts vs. p.r. functions

In Wittgenstein's terminology, formal concepts, such as identity in arithmetic or being tautologous in logic, are defined by the criterion that it is possible to identify whether their arguments satisfy the concept by syntactical manipulation. This syntactical manipulation must be performable by mechanical procedure in a finite number of steps. Thus, one might also relate the notion of formal concept to

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[12] Against Frascolla (1994), p. 13-20 and Frascolla (2000) who does not admit substitutions of η by Ω^{m}'η, but only of Ω^{m} for Ω. This forces him to introduce further definitions.

[13] Against Frascolla (1994), p. 16 and Frascolla (2000), p. 359 who maintains that $(\Omega$'Ω'$\xi)$ and Ω'Ω'ξ are "two different expressions with two distinct meanings". However, according to our interpretation, they are different signs, but the *same symbol*, without any significant difference of meaning.

the notion of decidability: A concept is formal iff it is decidable for each of its argument whether it satisfies the concept or not. According to Church's thesis, a function is decidable iff it is primitive recursive. From this, one might wonder whether Wittgenstein's notion of formal concepts is equivalent to that of p.r. functions. In this section, we will show that this is not the case.

That formal concepts are not equivalent to p.r. functions is evidenced by the fact that p.r. functions exist that are not formal concepts. This is due to the fact that their identity criteria are different. Formal concepts are identified by common syntactical features in a proper notation. In a proper notation, there are no different "modes of representation" of the formal concept. An argument satisfies a formal concept iff its expression can be converted to a symbol of this notation with the syntactical feature defining the formal concept. This ideal symbol is the representative of a class of equivalent expressions that are all reducible to the ideal symbol by syntactic manipulation. The "extension" of the formal concept is determined by the syntactical features of the ideal symbol; they are the identity criteria of a formal concept.

Of course, this does not mean that different expressions cannot satisfy the same formal concept. Formal concepts are applied to arguments represented by *different* expressions of a deficient symbolism that are converted to ideal symbols with *identical* syntactical features defining the formal concept in question. Formal concepts are not simply identified by their extension, but by a certain syntactical property that is common to the representation of their arguments in a proper notation. Thus, we call this conception "intensional". In contrast, p.r. functions rely on an extensional point of view: "p.r.ness is a feature of the function itself (identified extensionally), irrespective of how it happens to be represented to us" (Smith (2007), p. 88). Smith delivers two "dramatic examples" in order to emphasize this point – $fermat(n)$ and $julius(n)$, Smith (2007), p. 87f:

> $fermat(n) = n$ if there are solutions to $x^{n+3} + y^{n+3} = z^{n+3}$ (with x,y,z positive integers);
> $fermat(n) = 0$ otherwise.
>
> $julius(n) = n$ if Julius Caesar ate grapes on his third birthday;
> $julius(n) = 0$ otherwise.

$fermat(n)$ is p.r., because it is identical to $Z(n) = 0$ as "we know now – thanks to Andrew Wiles proof of Fermat's Last Theorem". $julius(n)$ is p.r. because it is either identical to $I(n) = n$ or to $Z(n) = 0$, although "we can't determine which function it is" (Smith (2007), p. 88). These two examples demonstrate that it does make sense to speak of p.r. functions, even though no routine is available for evaluating the functions. This is incompatible with Wittgenstein's notion of formal concepts. $fermat(n)$ and $julius(n)$ are not examples of formal concepts. Hence, there is no equivalence of p.r. functions and formal concepts. Even replacing $fermat(n) = 0$ with $Z(n) = 0$ does not make this a formal concept because it does not refer to expressions that are capable of syntactical manipulation. No formal concept is defined by some syntactical property, and no decision procedure is asked for to reveal some syntactical property by syntactical manipulation. By avoiding any expression that implies some logical or mathematical operation, one avoids the elements that constitute a formal concept of logic or mathematics.[14] Consequently, one must also deny that $Z(n) = 0$ is decidable unless it is represented in such a way that it contains arithmetic operations. Since no formal expressions are specified, there is nothing to decide. Moreover, because no operations are implied, there is nothing to compute. The notion of decidability cannot be separated from syntactical manipulation of expressions.

Note that $Z(n) = 0$ is meant to be a well-defined p.r. function, even if no routine is available to evaluate it. This shows that $Z(n) = 0$ is not only symbolically represented and paraphrased as a propositional function. Rather, it is meant to be, and to be used as, a propositional function. This does not only apply to the initial function $Z(n) = 0$; it also applies to p.r. functions in general. That they are actually used as propositional functions becomes evident in modern axiomatic systems, such as in PA. In p.r. functions of the form $\varphi(n)$, n is used as a *variable* that might be replaced by *any* number and might be bound by an universal or existential quantifier. The resulting formulae are truth functions – they are connected by logical particles and treated by the calculus of predicate logic. It is exactly this practice that Wittgenstein objected to throughout his life; the practice of treating arithmetic propositions as if they were propositions containing material concepts (and thus being true or false due to non-syntactical properties).[15] His intention, to

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[14] This does not mean that expressions have to explicitly contain some sign of operation. Numerals, e.g., are mathematical expressions although they do not contain a sign of some mathematical operation. However, the definition of natural numbers by $[x, \xi, \Omega\text{'}\xi]$ (cf. TLP 6.02) reveals that the proper notation of numbers implies an operation. A decision procedure that defines the formal concept of being a natural number properly requires converting numerals to expressions of the ideal Ω-notation. Hence, 3 is a number because, by the implicit syntactical rules of '3' this numeral has the multiplicity of $\Omega\text{'}\Omega\text{'}\Omega\text{'}\xi$, and this syntax defines natural numbers. Thus, '3' is a numeral (= sign of a number) due to syntactical rules and not due to some relation of reference to "the number 3". Likewise, the propositional formula P does not contain any logical particle, but its proper notation in the form $a - P - b$ reveals that ab-operations are applicable to this formula.

[15] cf. his life-long critique of set theory, illuminated in Rodych (2000). This critique is already

reconstruct arithmetic by purely syntactical means, is incompatible with a logical formalization of arithmetic and the use of predicate logic in axiomatic systems of arithmetic.

This is even true for axiomatic systems of primitive arithmetic that do not use quantifiers and logical particles, such as the system of arithmetic equations Z_{00} based on p.r. functions in Bernays and Hilbert's *Foundation of Mathematics*.[16] System Z_{00} still makes use of *free variables*, which becomes clear via the substitution rule.[17] This is due to using expressions of the form $f(x) = y$ as propositional functions. In contrast, according to Wittgenstein, 'x' is not used as a variable in arithmetic propositions such as $x + 2 = 4$, but as "the sign of an unknown quantity" (WVC, p. 109). It is possible to compute the value of the unknown quantity by an equivalence transformation: $x = 2$, that is, x is identical to 2 and not a variable for any number. $x+2 = 4$ does not mean $\exists x(x+2 = 4)$; it is not "true" of 2 and "false" of any other number. Instead, it is part of a calculus not dealing with truth and falsehood and the extension of functions; rather it deals with formal concepts satisfied by arguments that can be computed by syntactical means. This can be shown in the Ω-notation by converting $x + 2$ to Ω^{x}'Ω'Ω'η according to Def. + and Def. 2 and by converting 4 to Ω'Ω'Ω'Ω'η according to Def. 4. In order to transit from the former to the latter expression, one must substitute Ω^{x}'η by Ω'Ω'η, which is the Ω-expression of 2. Thus, the syntax of the proper notation reveals that x stands for the form Ω'Ω'η. Likewise, it can be shown by equivalence transformation that $x + 0 = x$ is satisfied by any number. $x+0 = x$ is not equivalent to $\forall x(x+0 = x)$. Instead, $x+0$ is expressed by Ω^{x+0} in the Ω-notation. This, again, is equivalent to Ω^{x}'Ω^{0}'η according to Def. + (cf. p. 86). According to Def. 0 (cf. TLP 6.02), this is equivalent to Ω^{x}'η, which is the Ω-expression of x. This shows that $+0$ is a redundant operation, similar to "is

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articulated in TLP 6.031 where Wittgenstein rejects the understanding of generality in mathematics in terms of universal quantification. Instead, he analyses generality in mathematics in terms of induction or, more precisely, by iterative applications of operations. The objection to applying truth operations and logical laws, as well as the objection to making use of logical quantification in the realm of mathematics, is a central topic in the middle period, cf. PR XII - XVIII, PG part II, section II . Later on, he still brands "the curse of the invasion of mathematics by mathematical logic" as "a method of writing [that] is nothing but the translation of vague ordinary prose" (RFM, V, §46).

[16] cf. also Goodstein (1945) to whom Marion (1998b), p. 107f. refers without noting that Goodstein as well as Skolem (1967) and Hilbert and Bernays deviate from Wittgenstein's intentions by still using p.r. functions in terms of propositional functions in the context of equations.

[17] $A(v) \vdash A(\tau)$. According to the substitution rule a free variable v can be replaced by any term τ.

true" in the case of bivalent propositions. According to Wittgenstein's analysis of arithmetic expressions, variables occur as exponents of operations and not in the argument places of functions. Variables are not defined by a certain domain over which functions assign values. Instead, they are part of expressions that calculate the number of applications of operations.

As our topic is not Wittgenstein's reconstruction of arithmetic, we will not discuss the extent to which his conception is applicable to non-trivial parts of arithmetic. Instead, we intend to show that Wittgenstein's position is incompatible with Church's thesis in this section. Plainly, from both the non-equivalence of p.r. functions and formal concepts, and the presumed equivalence of decidable formal properties with formal concepts, it follows that Wittgenstein rejects Church's thesis. This is due to his rejection of the underlying extensional understanding of p.r. functions. This understanding does not do justice to the fact that the decidability of a formal property is not a matter of facts corresponding to propositions that may be true or false. Understood correctly, a formal property is a matter of syntactical manipulation to the effect that the ideal symbolic representation of expressions in question allows identification of whether they satisfy the respective formal properties.

At stake is not only the adequacy of representing the concept of decidability, but also the proper analysis of formal properties. Are they adequately represented as propositional or characteristic p.r. functions? Or can they be traced back to purely syntactical properties identifying the formal properties? According to Church's thesis, the decidability of a formal property depends on there existing a characteristic decision function of the formal property in terms of a p.r. function. Bernays and Hilbert, for example, formalize the decidability of theorem existence in Z_{00} by the characteristic p.r. function $t(n)$. According to the standards of mathematical logic, formalizing theorem existence by a characteristic function is adequate because it *expresses* the formal property in question. $t(n) = 0$ is true according to the *standard interpretation* of the arithmetic language L for every natural number n iff n is the Gödel number of a theorem. This seems to be a plain and harmless formalization that harmonizes with the common understanding of arithmetic propositions as being either true or false. However, according to Wittgenstein, formalizing a decidable formal property this way is not adequate, precisely because it is represented by a characteristic function. Whether a formal expression has some formal property is conceived as a matter of fact, independent of the syntactical manipulations of the form of expressions. The seemingly harmless formalization is not harmless according to his point of view, because it is incompatible with his way of a purely syntactical account of formal properties.

From a Wittgensteinian point of view, a criterion of adequate formalization must not refer to either a *standard interpretation* or the presumed truth and falsehood of interpretations in terms of natural language statements. Any standard interpretation of this kind limits the space of possible interpretations. This is not compatible with Wittgenstein's view because to refer to an interpretation of formal language at all, one must refer to a formal semantics that is based on purely combinatorial possibilities that are unlimited.[18]

The traditional point of view does not explain to what extent statements on formal properties ϕ are not merely true statements that might come out to be false, but are rather necessarily valid statements. The traditional answer to this objection is the following. In the framework of a standard axiomatic system T of arithmetic, the formalization does not only *express* the formal property in question but it also *captures* it. Thus, in a decision function t of ϕ, the following is valid: For every natural number n, if n is ϕ, then $T \vdash t(n) = 0$, and if n is not ϕ, then $T \vdash t(n) = 1$. However, according to Wittgenstein's point of view, this only shifts the problem back to the mathematical axioms of T that are introduced by stipulation and cannot be proven on the basis of syntactic properties (cf. section 3.2). Wittgenstein's intention to analyze formal properties by syntactical features alone does not permit formalizing arithmetic concepts by propositional or characteristic functions within an axiomatic theory.

According to Wittgenstein's position, it is impossible to answer in advance the question of whether some decision problem is solvable because a formal concept is not definable properly independent of a decision procedure. This also applies to the formal concept of being a tautology in predicate logic. Unless tautologies cannot be identified by some common syntactical feature revealed by a decision procedure – as, for example, in the case of truth tables in propositional logic – no syntactical criterion is available that determines whether a certain formula is or is not "true under *all* interpretations" (cf. section 6.3). According to Wittgenstein's standard, this demonstrates that the traditional concept of being a tautology in predicate logic is still deficient. He demands syntactical criteria as basis for generating "extensions" like the class of models and counter-models of some *wff* in the realm of predicate logic, cf. PR, §186 as well as sections 3.1.6 and 6.3.4.

The critique of Church's thesis does not concern the possibility of some *arithmetic* representation of decidability, but rather the understanding and use of a

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[18] cf. the definition of classical semantics of predicate logic in section 6.3.1. For the problem of referring to some standard or suitable interpretation for the notion of adequate formalization within predicate logic cf. Baumgartner and Lampert (2008).

decision function in terms of a characteristic p.r. function. As a consequence of Wittgenstein's position, one must stipulate that, if an adequate formal representation of some formal property in terms of an arithmetic representation is available, it must make use of "signs of an unknown quantity" instead of variables. That means that it must be possible to compute x and y in an expression of the form $f(x) = y$ given x or given y. For example, in case of provability such an arithmetic representation would allow the computation of the Gödel number of a proof or disproof given the Gödel number of a formula. This already amounts to possessing two things: a precise arithmetic definition of the formal concept in question and a decision procedure by means of arithmetic computation. Like decidability, there is no sense in speaking of provability in advance, because the question of provability does not refer to some non-symbolic truth. Rather it refers to the possibility of certain syntactic manipulations, and these are only well-defined if the procedure can be carried out. According to Wittgenstein's point of view, any adequate formalization of provability is only available if one is able to present a proof or disproof. P.r. functions, as used in axiomatic calculi, for example in PA or PM, do not do justice to this claim. This follows from the formal representation of provability by use of p.r. functions. For example, this applies to Gödel's definition 46 in his incompleteness proof:[19] On the basis of a primitive recursive function xBy expressing "x is a proof of y" in PA or PM, he introduces an existential quantifier binding the variable y of Gödel numbers of proofs to represent the formal property of provability. Apparently, y is used as a variable and not as the sign of an unknown quantity. This way of formalizing provability is not based on the assumption that provability is only well-defined in a strict sense if a proof or disproof can be generated given the formula. However, Wittgenstein's proof conception claims just this. Quantifying over numbers of proofs contradicts his verdict of a logical formalization of formal properties. According to this argument a Wittgensteinian critique of metamathematical proofs and their methods does not refer to their informal paraphrases (as given by Gödel in the first line of his introduction to his paper from 1931)[20] in the first place, but to the use of

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[19] cf. Gödel (1931), p. 358: "46. Bew (x) $\equiv$ $(\exists y) yBx$." To be clear, $Bew(x)$ is not p.r. anymore.

[20] One standard objection against critics of Gödel's proof is that they do not take into account Gödel's formal proof but only refer to his informal proof sketch presented in his introduction of Gödel (1931), cf. Dawson (1988) according the reception of Gödel's incompleteness theorem. Gödel also replied to Wittgenstein's criticism under the terms of this standard objection, cf. Wang (1987), p. 48. Instead, we maintain that the origin of Wittgenstein's criticism relies in his rejection of propositional functions to formalize both, arithmetic as well as metamathemati-

propositional functions in order to represent formal concepts.

In section 3.1.5, we will argue that Wittgenstein's rejection of the use of propositional or characteristic functions to formalize formal properties does not only contradict Church's thesis, but also the method of diagonalization that is at the heart of undecidability proofs and transfinite set theory.

3.1.4 Arithmetic experiments

Before applying Wittgenstein's notions of operations and formal concepts to the programme of New Logic, we discuss the relations of p.r. functions, formal concepts, and operations. In this section, we address these by an example that underlines once more the radical nature of Wittgenstein's point of view and its incompatibility to a traditional standpoint, namely Wittgenstein's rejection of certain definitions as proper definitions of real numbers.[21] We will confine our discussion to the description of a number, P, as "the infinite binary fraction such that, at the nth place of the expansion, there occurs a '1' or a '0' according to whether n is prime or not" (PG, p. 275). The purpose of our discussion is to apply Wittgenstein's rejection of this definition to a critique of diagonalization in section 3.1.5 and traditional semantics in section 3.1.6.

At first sight, it seems strange that Wittgenstein rejects the apparently plain definition of P, because P is defined by a p.r. characteristic function. However, according to Wittgenstein's view, this is insufficient for a well-defined real number. This is due to his conviction that an "infinite series" is only definable by the unlimited application of an operation. In contrast to the definitions of even numbers, square numbers or Fibonacci numbers (cf. p. 80), a definition of all the primes in terms of an operation is not yet available. All that is available are definitions of infinite series of primes (though not necessarily all of them[22]) and decision procedures that calculate, for any number, whether it is prime or not. In the former case, a series of primes is defined by an operation, but it does not order

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cal properties. In this respect, we deviate from the controversy on Wittgenstein's discussion of Gödel's incompleteness theorem, cf. Rodych (2003) for an overview of this debate.

[21] This topic is studied thoroughly in the literature: cf. Da Silva (1993), Frascolla (1994), p.85-92, Marion (1998a), Rodych (1999) and Redecker (2006), cf. also Lampert (2008) according to Wittgenstein's understanding of primes and proofs of their infinitude.

[22] Euclid's proof of the infinity of primes, e.g., provides an inductive rule to generate an infinite series of primes without implying that all primes are part of the series: Given n primes, generate a new prime by decomposing $\prod_1^n p_i + 1$ in its prime factors. Starting with 2, this generates the series 2,3,7,43,13,53,....

the primes in a way that one can conclude that all primes are elements of the series. The latter case is usually demonstrated by defining the concept of primes as a p.r. function. The fact that primes are definable by a p.r. function, without providing any inductive rule or operation to generate the primes by iterative application, demonstrates once more that these two concepts are not equivalent. Commonly, primes are defined in modern mathematical logic as follows:

$$Pr(x) =_{def} (x \neq 1 \land \forall u \forall v(u \times v = x \to (u = 1 \lor v = 1))).$$

This defines a p.r. function (cf. Odifreddi (1989), p. 25), but it obviously does not define it in terms of an operation; instead it is defined in terms of a propositional function. This formalization of primes is incompatible with Wittgenstein's point of view for the following reasons:

1. It is incompatible with Wittgenstein's conception of arithmetic propositions as being valid on purely syntactically grounds.
2. It is incompatible with Wittgenstein's analysis of the concept "x is *a* prime" as a formal concept.
3. It is incompatible with Wittgenstein's standard of defining *the* primes by an operation.

These reasons will be explained in the following.

ad 1. According to the standards of mathematical logic, formalizing the formal concept of primes is adequate because it *expresses* the property of being a prime: $Pr(x)$ is true according to the *standard interpretation* of the arithmetic language L for every natural number n iff n is prime. Furthermore, in the framework of a standard axiomatic system T of arithmetic, the formalization not only *expresses* the concept of primes, it also *captures* the property of being a prime: For every natural number n, if n is prime, then $T \vdash Pr(n)$, and if n is not prime, then $T \vdash \neg Pr(n)$. As was already argued for on p. 91, Wittgenstein rejects expressing and capturing formal properties as standards of adequate formalization, as well as formalizing formal concepts using propositional functions. The traditional point of view does not explain in how "7 is a prime" is not a true statement that might have come out to be false, but a statement that is necessarily valid due to purely syntactical properties of expressions. Derivability from axioms introduced by stipulation is not a sufficient syntactical foundation of necessity.

ad 2. According to Wittgenstein's point of view, "x is prime" is a formal concept, and as such, it cannot be formalized by propositional functions, not even in terms of p.r. functions. A correct formal representation of the notion of a prime must demonstrate that it is impossible for 7 not to be a prime and for 6 to be a prime. A logical formalization of the concept of primes in the framework of an axiomatic theory cannot serve this purpose. Wittgenstein confronts the representation of primes as propositional functions, misled by surface grammar of ordinary language, with the idea of a “strict expression” within an ideal notation (PR, §159). The necessity of a prime to be a prime, and the impossibility of a non-prime to be a prime, can only be expressed adequately by an adequate syntactical representation of the numbers. Because of its form, this syntax must differentiate between primes and non-primes, making it impossible because of its syntactical properties to represent a prime as a non-prime and v.v. Contrary to ordinary language, the adequate formal expression renders transparent the fact that some number is prime by its syntax. Replacing the arabic numerals of numbers > 1 of the decimal system by the Ω-notation, and identifying the possibility of decomposing a number by groups of equal quantity $> \Omega$’, allows us to identify primes by the syntactical property of being represented by not more than one group of Ω’s within the Ω-notation:

2 Ω’Ω’ η
3 Ω’Ω’Ω’ η
4 Ω’Ω’ Ω’Ω’ η
5 Ω’Ω’Ω’Ω’Ω’ η
6 Ω’Ω’ Ω’Ω’ Ω’Ω’ η
7 Ω’Ω’Ω’Ω’Ω’Ω’Ω’ η
8 Ω’Ω’ Ω’Ω’ Ω’Ω’ Ω’Ω’ η
9 Ω’Ω’Ω’ Ω’Ω’Ω’ Ω’Ω’Ω’ η
...

Thus, according to Wittgenstein’s conception, deciding whether a number is prime or not means to possess a procedure that converts arabic numerals to expressions of Ω-notation decomposed in equal parts. According to this point of view, the concept of a prime depends on the mode of representation in a proper notation. The numbers 2, 3, 5, 7, 11, and 13 are primes because they share the syntactical feature of not being represented by several groups of Ω’s in a proper notation.

ad 3. Although it is decidable whether some number is prime, and although the concept of being *a* prime is a formal concept, Wittgenstein still rejects the definition of P as a proper definition of a real number. The reason is that, by the decision procedure, no operation is defined that would allow primes to be generated iteratively. According to Wittgenstein's standards, only an operation can define an "infinite series". In particular, the operation must not only generate new primes from previous primes, as Euclid's inductive rule, but must further order primes by some internal relation such that no prime is left out.[23] Only such a definition is able to serve as a definition of "the general form of the primes" (cf. PR §159). Unless such a definition is available, the provided definition of P does not define an infinite binary fraction. This is incompatible with the notion that a characteristic p.r. function of natural numbers defines an infinite series of binary fractions. However, this criterion is based on an extensional point of view treating infinity as an extension – some *number* that is greater than any "finite" number. According to Wittgenstein's point of view, this is another misconception that, misled by the surface grammar of ordinary speech, treats the word "infinite" as a number word instead of doing justice to the actual use of the word. Such a treatment does not refer to the magnitude of an extension, but rather provides a general possibility for progression (cf., e.g., WVC, p. 227-231).

However, independent of the interpretation of P as p.r. function, one might still suggest that P defines an infinite series. One simply has to go through the natural numbers and decide for each number whether it is prime or not. The series of natural numbers is well-defined by an operation in Wittgenstein's terms, and it is decidable, whether a natural number is prime or not. Thus, one might assume that this is a well-defined method to proceed ad infinitum. However, it is exactly this conception that Wittgenstein is attacking with his rejection of P. "[Dealing] with the numbers one by one" he says "doesn't lead to the totality" (PR, p. 146). Instead, it only determines finite sequences of a series but never the *complete* series. Although the procedure is not limited to some special extension, it essentially refers to some finite *extension*. There is no general rule in advance that necessarily generates primes by iteration. Thus, the problem arises that in order to *determine* primes one has actually to *apply* the procedure. However, any actual *application* of the procedure must be finite. Thus, the effective calculation of single members of a series cannot determine an infinite totality. Even if one concedes that it has a meaning how to go on applying this kind of procedure "ad

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[23] Euclid's rule does not order the primes by some internal relation. This is because $\prod_1^n p_i + 1$ does not necessarily generate a new prime. Only the decomposition of the result of this product in primes necessarily generates a new prime.

infinitum," it thereby does not have a distinct meaning how the series of primes goes on "ad infinitum". This becomes clear by the fact that this procedure does not imply that there is no greatest prime number. Without referring to some inductive rule to generate primes, it may well be that no further number comes out to be prime. The dots in "2,3,5,7,11 . . ." are not well-defined.

In contrast, Wittgenstein claims that any well-defined infinite series of numbers must be definable by constant variations of the *syntactical properties* of the number symbols themselves. Not by *descriptions*, which refer to some properties not revealed in the syntax of number symbols, but only by *operations*, which define variations of syntactic properties, it is definable properly how a series of numbers goes on ad infinitum. Referring to syntactic variations allows for defining how to go on by iteration. Thus, instead of having to decide first whether a possible new member of a series satisfies some property, the following member is generating directly from previous ones. This way of defining how to go on ad infinitum by operation is the only possible one according to Wittgenstein, cf. TLP 5.2523:[24]

> The concept of successive applications of an operation is equivalent to the concept 'and so on'.

Wittgenstein also expresses his requirement of defining infinite series in terms of operations by opposing it to what he calls "arithmetic experiments", cf. PR, §190[1]:

> In this context we keep coming up against something that could be called an 'arithmetical experiment'. Admittedly the data determine the result, but I can't see *in what way* they determine it. (cf. e.g. the occurrences of 7 in π.) The primes likewise come out from the method for looking for them, as the results of an experiment. To be sure, I can convince myself that 7 is a prime, but I can't see the connection between it and the condition it satisfies. – I have only found the number, not generated it.
> I look for it, but I don't generate it. I can certainly see a law in the rule which tells me how to find the primes, but not in the numbers that result. And so, it is unlike the case $+\frac{1}{1!}, -\frac{1}{3!}, +\frac{1}{5!}$ etc., where I can see a law *in the numbers.*
> I must be able to write down a part of the series, in such a way that you can *recognize* the law. That is to say, no *description* is to occur in what is written down, everything must be represented.

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[24] In PR, §189 Wittgenstein uses the term "induction' instead of "operation": "The true nature of real numbers must be the induction. What I must look at in the real numbers, its sign, is the induction. – The '*So*' of which we may say 'and so on'."

> The approximations must themselves form what is *manifestly* a series.
> That is, the approximations themselves must obey a law.

Generating a series of numbers by deciding one by one whether it satisfies some property is what Wittgenstein calls an "arithmetic experiment". It obeys mechanical rules, constructing finite sequences of a series of numbers with a certain property, but it lacks the essential feature of a well-defined infinite series. It does not have the form of an operation that defines some *internal* relation based on syntactic properties of number symbols.

On the first sight, Wittgenstein's condition to define infinite series in general, and real numbers in particular, by operations seems to be by far too strict. According to this condition not only exotic numbers such as P are exposed as pseudo-irrationals but also most common irrationals such as $\sqrt{2}$ seem not to be in conflict with this condition. In fact, no law can be recognized in its decimal expansion. From Wittgenstein's criteria, it follows that the dots in "1.414213 . . ." are not well-defined. Thus, for example, Redecker concludes that Wittgenstein's condition that the law of a series must be recognizable in its elements is by far too strong.[25]

However, Wittgenstein is well aware of this problem. He, in fact, explicitly draws the conclusion, that "the procedure of extracting root 2 *in the decimal system* is an arithmetic experiment, too"(MS 107, p. 91, translated by myself). This, indeed, means that he rejects to defining $\sqrt{2}$ by the effective process for generating rational numbers in the decimal system whose squares approximate more and more to 2 (against Frascolla (1994), p. 92). He generally rejects any definitions of numbers by descriptions.[26] However, the failure to satisfy Wittgenstein's standards of defining $\sqrt{2}$ by referring to the decimal system, does not mean that $\sqrt{2}$ is not definable properly according to Wittgenstein's standards. Wittgenstein immediately goes on to clarify what consequences must be drawn from rejecting the procedure of extracting root 2 *in the decimal system* (MS 107, p. 91, translated by myself):[27]

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[25] Redecker (2006), p. 212 discussing Frascolla (1994), p. 91f who also deals with the problem that the decimal expansion of $\sqrt{2}$ seems to be as ill-defined as P according to this criterion.

[26] cf. PR §196: "Numbers as the result of an arithmetical experiment, and so the experiment as the *description* of a number, is an absurdity.
The experiment would be the description, not the *representation* of a number."

[27] cf. also MS 107, p. 123, my translation: "The law of a sequence can be comprehensible immediately in the decimal system, e.g. 0.1010010001 etc, or not; then it must be comprehensible in another system of expression."

> However, this only means that this procedure is not completely essential to $\sqrt{2}$ and a representation must exist that makes the law recognizable purely.

Later, he then refers to the notation of *continued fractions* that makes it possible to define $\sqrt{2}$ by an operation (MS 107, p. 126, translated by myself):[28]

> [...] in $\frac{1}{2}, \frac{1}{2+\frac{1}{2}}, \frac{1}{2+\frac{1}{2+\frac{1}{2}}}$ one can recognize the law that one cannot recognize in the decimal development.

The same also applies to other prominent irrationals such as e or π. Whereas the dots "..." are not well-defined in case of the decimal fraction, they are well-defined in the notation of continued fractions:

name	**decimal number**	**continued fraction**	**shorthand notation**[29]
$\sqrt{2}$	1,414213 ...	$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}$	$[1; 2,2,2,\ldots]$
e	2,718281 ...	$2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\ddots}}}}}$	$[2; 1,2,1,1,4,1,1,6,\ldots]$
π	3,141592 ...	$\cfrac{4}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{\ddots}}}}$	

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[28] By Heron's procedure, $\sqrt{a}$ can be generated by iterative application of the operation $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$. This is another possibility to define square roots by an operation.

[29] The shorthand notation is only available for continued fractions having only 1 in the numerator. As in π the values of the numerators differ, so no shorthand notation of this form can be given.

Contrary to the decimal fractions, one can "recognize a law" in the continued fractions and define them by operations. Defining real numbers by operations essentially depends on the syntax of a notation used to represent numbers. This also explains why $\overset{5\to 3}{\sqrt{2}}$ (= the number that is identical to the decimal fraction of $\sqrt{2}$ but replaces any 5 by 3) is ill-defined, whereas $\overset{5\to 3}{1/7}$ (= the number that is identical to the decimal fraction of $1/7$ but replaces any 5 by 3) is not ill-defined (cf. PR, §183, PG, II 42). As $\sqrt{2}$ is not definable by an operation that refers to the decimal notation, $\overset{5\to 3}{\sqrt{2}}$ is not definable by an operation. On the contrary, $\overset{5\to 3}{1/7}$ is definable by an operation that refers to the decimal notation as $1/7$ is periodic: $1/7 = 0{,}\overline{142857} = [0{,}132857, \xi, \xi 132857]$, and hence $\overset{5\to 3}{1/7} = 0{,}\overline{142837} = [0{,}142837, \xi, \xi 142837]$. Thus, contrary to Frascolla (1994), p. 89-92, Da Silva (1993), p. 94 and Rodych (1999), p. 283, 285, we do not regard further criteria as necessary for defining real numbers in addition to the criterion to define them by laws in terms of operations, in order to explain why $\overset{5\to 3}{1/7}$ is well-defined whereas $\overset{5\to 3}{\sqrt{2}}$ is not. Also, contrary to most of the literature, we maintain that this criterion is sufficient to rule out all the definitions of "pseudo-irrationals" Wittgenstein rejects. The view that this is not so basically relies on the erroneous presumption that operations are equivalent to, and do not amount to more than, p.r. functions or some effective procedure to calculate single members of a series. However, all pseudo-irrationals Wittgenstein mentions are defined by *descriptions* that cannot be replaced by operations. They all make the expansion of an infinite series dependent on some property that cannot be reduced to variations of syntactic properties of a proper mode of representation. Wittgenstein's claim to define real numbers by operations is the demand of a proper representation, MS 107, p. 89, my translation:

> I want a representation of the real number that reveals the number in an induction such that I have herewith the only proper, unambiguous symbol.

Wittgenstein's demand of a proper representation, in turn, is the demand of an absolute clarity that can only be achieved in logic and arithmetic by expelling any reference to non-symbolic properties, cf. PG, II, p. 480:[30]

> Only what I *see* is a law; not what I *describe*. That is the only thing standing in the way of my expressing more in my signs than I can understand.

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[30] cf. also MS 107, p. 53, my translation: "It cannot be that I possess a proper representation that I do not understand. Instead, in this case something must be out of order."

"Irrational sequences" in the sense of sequences not obeying any law, are rather a symptom of a deficient notation according to this view than a reason for believing in the existence of numbers of which we can only know finite parts without ever grasping how they extend to infinity. Wittgenstein's view of real numbers is a further example of his intensional conception of arithmetic and logic, which is essentially connected to the construction of a proper notation that alone makes it possible to define formal concepts and operations. His conception cannot be judged without considering modes of representation realizing it. From a traditional point of view, one might object that his conception is in conflict with the common understanding of mathematics and logic. From Wittgenstein's point of view common mathematics and logic is misguided by inadequate formalisms and must be clarified first by inventing adequate symbolic representations.

3.1.5 Diagonalization

In this section, we will point out that Wittgenstein's rejection of defining infinite totalities with functions instead of operations is incompatible with the method of diagonalization as it is applied in metamathematics and transfinite set theory. In particular, we will argue that his point of view contradicts Cantor's, as well as Church's, theorems.

Cantor's proof of the in-denumerability of real numbers rests on the definition of a diagonal number $\mathcal{D}$[31]: Given an enumeration of real numbers of interval $(0,1]$ in terms of infinite binary sequences, the n'th digit of $\mathcal{D}$ is 0 / 1 iff the n'th digit of the n'th binary sequence is 1 / 0. From this, it follows that $\mathcal{D}$ cannot be part of the enumeration of the binary sequences, as assuming $\mathcal{D}$ is the m'th binary sequence, the m'th digit of $\mathcal{D}$ is 0 iff it is 1. As it is assumed that $\mathcal{D}$ is a well-defined number, it follows that a real number exists that is not part of the enumeration of reals. Thus, the assumption that the enumeration enumerates all real numbers is reduced to absurdity. Consequently, the reals are in-denumerable.

Wittgenstein rejects the assumption that the so called "diagonal number" is a well-defined number.[32] According to his standards, diagonal numbers are a special case of arithmetic experiments because it must be decided one by one whether the next digit is 0 or 1. Only finite sequences of $\mathcal{D}$ are well-defined by this procedure. However, as long as no operation is defined that allows one

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[31] Strictly speaking, it would be more accurate to call $\mathcal{D}$ the "antidiagonal number" and its complement "diagonal number". However, we follow common terminology and refer to antidiagonal sequences as diagonal numbers.

[32] cf. RFM II, §1-22 and Redecker (2006), chapter 2 - 4 for a detailed analysis.

to generate the following digits from the preceding ones, no proper definition of an infinite binary sequence is at hand. One refers to an “infinite totality” with a description in terms of a characteristic function expressed in ordinary prose, rather than providing an operation in terms of an arithmetic law referring to some arithmetic notation.

As $\mathcal{D}$ is not well-defined in terms of an infinite sequence, there is no reason to believe in transfinite numbers or to state that “the number of the real numbers is greater than the number of natural numbers”. Wittgenstein even rejects that it makes sense to speak of “the” real numbers. The reason for this is the same as in case of his rejection of “the primes” (in contrast to “a prime”): As long as no operation is available that allows one to generate the respective totalities by iteration, the terms are not well-defined. Real numbers do not constitute “a system”. This is the real difference to the natural and the rational numbers. However, to conclude that we have well-defined totalities of a different cardinality in both cases is a misguided expression of this difference: “There is no system of irrational numbers – but also no super-system, no ‘set of irrational numbers’ of higher-order infinity.” (RFM II, §33)

From Wittgenstein’s point of view, complements of diagonal numbers are ill-defined, too.[33] $\overline{\mathcal{D}}$ is defined as follows: Given an enumeration of real numbers of interval $(0{,}1]$ in terms of infinite binary strings, the n’th digit of $\overline{\mathcal{D}}$ is 0 / 1 iff the n’th digit of the n’th irrational is 0 / 1. In this case, no contradiction follows from the assumption that $\overline{\mathcal{D}}$ is part of the enumeration. All that follows from assuming that $\overline{\mathcal{D}}$ is the m’th binary digit is that we cannot know whether the m’th digit is 0 or 1. However, according to the traditional point of view, $\overline{\mathcal{D}}$ exists and its m’th digit is either 0 or 1, independent of the impossibility of determining the m’th digit of $\overline{\mathcal{D}}$. This illustrates the extensional, platonistic understanding of numbers. In contrast, by Wittgenstein’s standards, it does not make sense to speak of a number unless it is entirely determined by its definition. This condition is not satisfied if it is not possible to develop a sequence to any arbitrary extension.

Wittgenstein’s critique does not only apply to diagonal numbers, but also to the diagonal method in general, which is at the heart of undecidability proofs. This, we illustrate by Bernays’ and Hilbert’s undecidability proof of the axiomatic system Z_{00} of equations containing p.r. functions (cf. Bernays (1970), p. 443). Under Church’s thesis, the decidability of provability of Z_{00}-formulae must be captured by a decision function, $t(n)$, assigning to their Gödel numbers the val-

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[33] Complements of diagonal numbers and of diagonal functions are also relevant for metamathematical proofs, cf. Smith (2007), chapter 5.2.

ues 0 or 1 according to their provability. As $t(n)$ must be p.r., it must be part of Z_{00} if the provability of Z_{00}-formulae is decidable. Diagonalization comes into play by defining a function, $s(k)$, whose values are the Gödel numbers of those formulae that are generated by replacing the variable a in a Z_{00}-formulae $\mathcal{A}$ with Gödel number k by k. The diagonalization of formula $t(s(a)) = 1$ with Gödel number l is $t(s(l)) = 1$. We assume that this formula has Gödel number m. Based on the assumption of Church's thesis, this leads in case of $t(m) = 1$ to the same situation as in case of diagonal numbers (cf. Lampert (2005b) for a detailed reconstruction): the value of $t(m)$ is 0 iff it is 1. According to the interpretation of the function as a decision function, that means that the formula with Gödel number m is provable iff it is not provable. As in the case of diagonal numbers, it is concluded from this contradiction that a Z_{00}-formula exists that is not decidable. According to Wittgenstein's standards, however, the inconsistency demonstrates that the totality of the decidable Z_{00}-formulae, differentiated according to their provability, is not well-defined by the p.r. function $t(n)$. The reason for this is the same as for diagonal numbers: instead of defining the totalities by an operation, they are defined by a characteristic function expressed in ordinary prose instead of some pure arithmetic formula. As Bernays and Hilbert base their proof of the undecidability proof of first order logic on the undecidability of Z_{00}, similar to how Church referred to the undecidability of arithmetic system L in his undecidability proof of first order logic in Church (1936), their proof of Church's theorem is as inconclusive as the proof of undecidability of Z_{00}. This, of course, does not mean that Z_{00} is decidable. From Wittgenstein's point of view, it cannot be assumed that provability, or more precisely, "*a* provable Z_{00}-formula", is well-defined as long as no decision procedure is available and that "*the* provable formulae of Z_{00}" are well-defined as long as no operation is available for generating the provable formulae.

Similar objections apply to the proof of the undecidability of first order logic on the basis of Turing's thesis. This proof is based upon the insolvability of the halting-problem for Turing machines, which, in turn, refers back to the incomputability of a diagonal function $d(n)$ by a Turing-machine (cf. Boolos (2003), chapter 4). Given an enumeration of Turing machines, the value of $d(n)$ is 2 if the n'th Turing machine computes 1 for input n and 1 otherwise. As in the case of Cantor's proof of the in-denumerability of real numbers, the proof of the incomputability of $d(n)$ deduces a contradiction from the assumption that $d(n)$ is itself computed by a Turing machine. Given that this machine is the m'th in the enumeration of Turing machines, it follows that $d(m)$ is 1 iff it is 2. According to Wittgenstein's standard, this inconsistency is caused by the misconception of

$d(n)$ as a “perfectly genuine total function” (Boolos (2003), p. 37). Instead, far from proving the incomputability of the diagonal function, the so-called proof demonstrates that the diagonal function is no “perfectly genuine total function” that determines its values for all natural numbers. Diagonal functions, as well as diagonal numbers, are only well-defined for finite enumerations of sequences of functions not containing the diagonal function in question. They are unable to determine an infinite totality of values as they refer to the process of deciding, one by one, what the next value is. They do not provide any operation that determines the values in advance by defining a rule to generate succeeding values from previous values. As the undecidability proof of first order logic on the basis of the Turing-thesis rests on the proof of the incomputability of some diagonal function $d(n)$, it is also questioned by Wittgenstein’s rejection of diagonal functions. Furthermore, by formalizing operations of Turing machines by predicate logic, it also becomes obvious that the proof of Church’s theorem on the basis of Turing’s thesis rests on the lack of formal distinction between operations and propositional functions (cf. Boolos (2003), chapter 11).

Thus, proofs of Church’s theorem rest on an extensional understanding of arithmetic functions admitting diagonal functions. They have no probative force beyond this understanding. They are more of an *expression* of the paradigm of “Old Logic” than a proof of one of its fundamental theorems. Gumanski (1986) also denies the method of diagonalization and rejects Church’s theorem (cf. also Gumanski (2000)). However, like critics of diagonalization such as Kronecker, Brower, Poincare, and others, he does not base his rejection of diagonalization on the negative account of defining infinite totalities by genuine total characteristic functions and the positive account of defining infinite totalities by operations, as Wittgenstein does.

It is the uncritical use of propositional and characteristic functions, and the reasoning based on these functions that meta-mathematical proofs still have in common with classical paradoxes. Richard’s paradox, for example, has the same structure as Cantor’s proof or the proof of the incomputability of $d(n)$ (cf. Lampert (2007)). Referring to an enumeration of definitions of sets of numbers, the set of Richard numbers is defined as follows: n is a Richard number iff n is not member of the set defined by the n’th definition. As this definition seems itself to define a set of numbers, it is part of the enumeration. Yet, this yields a contradiction: given the definition of Richard numbers is the m’th in the enumeration of definitions of sets of numbers, m is a Richard number iff m is not a Richard number. However, in this case, concluding that the definition of Richard numbers is not part of the enumeration of definitions of sets of numbers is to reject the definition itself. Thus, unlike the case of Cantor's proof or the proof of the incomputability of $d(n)$, one must reject the definition of a diagonal function in the case of Richard's Paradox even according to the traditional argumentation. Usually, this is justified by the fact that, contrary to the definition of diagonal functions in transfinite set theory or meta-mathematics, the definition of Richard numbers confuses meta- and object-language, as it refers to the notion of definability in the definitions (cf. Boolos (2003), p. 342, Nagel (2001), p. 89f., footnote 27 and Delong (1971), p. 256). According to the common view, this is the reason why Richard's Paradox is a fallacy, whereas diagonalization does not cause fallacies in the case of meta-mathematics or transfinite set theory. However, Boolos (2003), p. 21 suggest "replac[ing] each definition by the set it defines" and they mention that Richard's Paradox is isomorphic to Cantor's proof of the in-denumerability of all sets of natural numbers. Thus, it is dubious to treat them not alike. From a Wittgensteinian point of view, Richard's Paradox reveals the deficiency of all the purported proofs based on diagonal functions, namely, to define infinite totalities by means of functions expressed by ordinary language. The possibility of all of these fallacies is only ruled out by claiming that infinite totalities must be defined by operations determining the form of symbols.

Similar considerations hold for the Liar Paradox. As long as the grammatical predicate "x is false" is analyzed as a propositional function, the possibility of diagonalization is not excluded by the interpretation of the real syntax of this expression. On the contrary, the analysis of "x is false" in terms of an operation, the operation of negation (TLP 5.512), rules out the possibility of diagonalization, and thus, the possibility to construct the Liar Paradox using the syntax of a proper notation. Negation is a truth operation that is applied to the poles of bivalent propositions (propositions capable of truth and falsehood). Bivalent propositions are built up by propositional functions. Thus, "x" in "x is false" is not a variable open for paradox substitutions according to diagonalization, but rather the base of a truth operation that syntactically presumes a proposition containing a propositional function with two poles. This condition is not satisfied in case of "This proposition is false". This expression does not contain any propositional function providing poles to which the operation "is false" could be applied. It seems to be of the same form as "This banana is yellow". However, an analysis of its logical form must not refer to its grammatical form, but to its actual use in ordinary language. However, contrary to "This banana is yellow," the expression "This proposition is false" does not have any use in ordinary language. Any analysis of "x is false" as a propositional function erroneously concludes, from the fact that "x is false" is a predicate according to English grammar, that it has to be used as a propositional function. This also applies to Tarski's "solution" of the Liar Paradox, which involves distinguishing object- and meta-language and interpreting "x is false" as a semantic predicate of meta-language (cf. Tarski (1996)). From Wittgenstein's point of view, it does not suffice to distinguish the structure of meta-mathematical proofs from the structure of paradoxes by referring to the strict differentiation between object- and meta-language in meta-mathematical proofs. On the contrary, this argument demonstrates that the spring of the paradoxes is still bubbling; the analysis of formal properties and operations in terms of propositional functions.

As long as one sticks to an analysis of language that treats the grammatical form of propositions as a criterion of their real form, to an analysis that treats mathematical reasoning as some form of logical deduction and every use of "all" or "exists" as some form of logical quantification, Wittgensteinian philosophy "is a battle against the bewitchment of our intelligence by means of our language" (PI, §109). Similar to paradoxes, meta-mathematical proof methods are only forcible as long as one does not see any alternative analysis that does justice to the actual use of propositions by elaborating different formal representations for different kinds of propositions.

3.1.6 The problem of semantics

The semantics of predicate logic also refers to a totality: the totality of interpretations $\Im$. Contrary to propositional logic, one must refer to a domain with an infinite number of objects in predicate logic. Consequently, the totality of interpretations, $\Im$, constitutes an "infinite totality". The task of semantics is to differentiate between those interpretations that are models and those that are counter-models of a *wff* A. For interpretations with a finite domain, this task can be solved algorithmically according to classical semantics by evaluating single interpretations $\Im$.[34] However, as in arithmetic experiments, dealing with the interpretations one by one will never "lead to the totality" if one has to consider $\Im$ with infinite domains. Thus, not only in defining real numbers, but also in the semantics of predicate logic, there exists the problem of how to determine an infinite totality. According

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[34] cf. section 6.3.1 where we define classical semantics in detail. A computer programme for evaluating $\Im$ with finite domains is available under http://philoscience.unibe.ch/logik.html. Note that we define the semantics of predicate logic without referring to some "standard interpretation" in terms of ordinary language. Instead, we define classical semantics by purely combinatorial means. In section 6.3.2, we confront this version of classical semantics schematically with our conception of New Semantics.

to Wittgenstein's standard, the traditional notions of semantics are as deficient as those of arithmetic experiments: the totality of models of a *wff* A, $\Im_T$, and the totality of counter-models of a A, $\Im_F$, are determined by examining interpretations one by one to identify models and counter-models of A. However, by this method only a finite number of models or counter-models can be actually determined.

This critique applies to the definition of tautologies in first order logic (FOL). A *wff* A of FOL is a tautology iff *all* its interpretations $\Im$ are *models*. Thus it is referred to an infinite totality of models. Consider the following attempt to define a real number T by a characteristic function referring to models and counter-models of a *wff* A: T is the binary fraction $a_1.a_2a_3\ldots$ with $a_n = 1$ if the n'th $\Im$ is a model of A and $a_n = 0$ if the n'th $\Im$ is a counter-model of A.[35] A *wff* A is a tautology iff $T = 1.\bar{1}$. However, T is just another pseudo-irrational defined by a description in terms of a characteristic function. *wff*s having counter-models only in case of infinite domains are good examples to illustrate the impossibility to define the totality of models this way (cf. p. 299). The method of approximation produces only 1's in the sequence of T in this case although the respective *wff*s are no tautologies. According to Wittgenstein's standards any proper definition of tautologies in FOL must refer to syntactic properties of a proper representation of the respective *wff* that allow one to identify tautologies as tautologies and non-tautologies as non-tautologies. This, in turn, implies that a proper definition of tautologies in FOL is only available if FOL is decidable by means of a syntactic procedure that converts *wff* to their proper presentation in a finite number of steps.

From a Wittgensteinian point of view, the totality of $\Im_T$ and $\Im_F$ of a *wff* A must be generated iteratively, without determining whether each interpretation is a model or a counter-model.

problem of semantics: The problem of semantics is generating the totality of models $\Im_T$ and counter-models $\Im_F$ of a *wff* without examining the single interpretations $\Im$ one by one.

We intend to solve this problem in section 6.3. Like Wittgenstein's intensional characterization of real numbers as laws, our solution essentially rests on proper notation. By converting *wff*s to ab-symbols, we generate expressions that enable us "to recognize a law" by their syntactical features. We will demonstrate this by showing how $\Im_T$ and $\Im_F$ of a *wff* A can be generated iteratively given the ab-symbol of A. By means of ab-symbols, we intend to realize an intensional

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[35] We presume that $\Im$s can be generated one by one starting with a domain with only one object c_1, cf. section 6.3.1.

conception of logic that provides *syntactical criteria* that identify the arguments satisfying the formal concepts "is a model of A" and "is a counter-model of A" without examining interpretations one by one.

3.1.7 The problem of implication

Wittgenstein defines another totality in the *Tractatus*: the totality of all truth functions. This totality he calls "the general form of a truth-function" (TLP, remark 6) and defines it by the operation $N(\overline{\xi})$:

$$[\overline{p}, \overline{\xi}, N(\overline{\xi})]$$

$\overline{p}$ stands for all atomic propositions, ξ stands for any set of propositions, and $N(\xi)$ stands for the negation of all the propositions making up ξ.[36] As long as one applies this definition to propositional logic and to a finite number of atomic propositions, it is equivalent to constructing all truth functions by the Sheffer stroke in terms of "neither ... nor ..." (cf. TLP 5.51, CL, p. 122 and p. 126). However, contrary to Sheffer, Wittgenstein also applies his definition to predicate logic. This becomes clear in his introduction of quantification in TLP 5.52:

> If ξ has as its values all the values of a function fx for all values of x, then $N(\overline{\xi}) = \sim (\exists x).fx$.

This implies that ξ might also range over an infinite number of propositions of form fx.[37] However, fx is a propositional function. Hence, this infinite number of propositions is determined by a propositional function and not by an operation, cf. TLP 5.501. This contradicts Wittgenstein's own claim to refer to any infinitude by operations alone. Wittgenstein later called this falling behind his own claim the "biggest mistake he had made in the *Tractatus*" (Wright (1982), p. 151). By specifying the totality of propositions, he treats "infinity as a number, and suppos[es] that there can be an infinite number of propositions" (*Lectures 1930-32*, p. 119, cf. *Lectures 1930-33*, p. 298f., PG part II, II 8). He analysed quantified propositions as truth functions of an infinite number of propositions that "could be enumerated, though we were unable to enumerate them" (*Lectures 1930-33*, p.

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[36] cf. Russell's introduction to TLP, p. xv, Graßhoff (2004), p. 237, cf. also TLP 5.5 - 5.51 and CL, letter 68, point (9) referring to Russell's question according remark 6, cf. letter 67, cf. also Anscombe (1996), chapter 10 and Frascolla (1994), p. 2.

[37] cf. CL, p. 126: "A general prop[osition] is A truth-function of *all* PROP[OSITION]S of a certain form."

298f.). Thus, although Wittgenstein did sharply distinguish between operations and functions in the TLP and reduced formal series to the iterative applications of operations, he still admitted an extensional understanding of infinity in the realm of predicate logic. This account of quantification the TLP has in common with traditional logic.

This mistake of the TLP must be eliminated from the conception of New Logic. Wittgenstein's New Logic, as we conceive it, is primarily part of the programme of a purely syntactical foundation of logic that is opposed to mathematical logic. This makes it systematically and historically interesting. As we are concerned with the possibility of a systematic elaboration of this conception as a consequence of his radical distinction between operations and propositional functions, we will abstain from presenting any account of quantification not serving this purpose. However, the mere intention to do so is insufficient for a systematic elaboration of New Logic, because *realizing* Wittgenstein's conception in the realm of predicate logic is the question in the first place. Yet, our elaboration of New Logic intends to demonstrate that, by his ab-notation and its use of quantifiers as an irreducible part of this ab-notation, Wittgenstein did possess another means of quantification that must be preferred to his TLP-treatment of quantification.[38] Thus, the "biggest mistake" in his conception of logic in the TLP was to abstain from his previous "new notation" which is an essential part of New Logic. One can only speculate on his reason for doing so; there is not the slightest hint for any criticism of his ab-notation. On the contrary, he seemed quite confident that this notation solves all problems of predicate logic. However, treating quantification as an irreducible part of a proper notation certainly does not allow logic to be treated as scantily as TLP does. In particular, the definition of the general form of truth function cannot simply refer to atomic propositions and their negations. From this, it follows that one must seek a new definition of the general form of a truth function.

problem of the general form of truth function: The problem of the general form of a truth function is defining the totality of truth functions expressible in predicate logic by iterative application of operations.

This problem is solved in this book by generalizing the concept of atomic propositions, or, more precisely, by generalizing the concept of literals (negated

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[38] Thus, unlike Landini (2007), p. 129-146 we do not consider to solve the equivalence problem on the basis of the "N-operator notation". For this reason, the discussion of the attempt to base the theory of quantification on the N-operator by Fogelin (1987), Geach (1981), Kibéd (1993) and Landini (2007) does not concern our elaboration of the programme of New Logic.

and non-negated atomic propositions). We obtain this generalized concept from the notion of a complex pole in Wittgenstein's ab-notation. In the realm of propositional logic, these complex poles correspond to assigning truth values to propositional variables in the left part of truth tables, or to literals such as P or $\neg P$ in disjunctive normal forms of propositional logic. In the realm of entire predicate logic, they correspond to what we call "closed structures". Contrary to literals of propositional logic, closed structures and complex poles may contain quantifiers. Closed structures are predicate formulae with minimized scopes of quantifiers (cf. section 6.1.1). By defining an equivalence transformation for predicate formulae, we will demonstrate that any truth function in predicate logic can be expressed by groups of complex poles in the ab-notation (pole-groups) or by a disjunction of conjunctions of closed structures ($\bigvee \bigwedge cs$ in short) in predicate logic (cf. section 6.2). Closed structures can be defined inductively (cf. p. 228; their totality can be generated by iterative application of operations. As any truth function of closed structures, in turn, can be expressed by applying the operation $N(\xi)$ or Sheffer's operation to closed structures, the problem of the general form of truth functions is solvable by referring to closed structures, instead of referring to atomic propositions of propositional logic. Closed structures replace the concept of atomic proposition in our elaboration of New Logic.

However, from this, a more fundamental problem arises, namely the problem to specify the internal relation of predicate formula by iteration. Internal relations of predicate formulae are implication, contrariety and and sub-contrariety. The relations of contrariety and sub-contrariety can be defined by implication and contradiction. A formula A is contradictory to a formula B if it is equivalent to $\neg A$. Equivalence is again defined by the relation of implication. Thus, the problem to specify internal relations between predicate formula is reducible to specify their relations of implication. Within the conception of New Logic the specification of implications between predicate formula is reduced to the specification of implications between $\bigvee \bigwedge cs$ or pole-groups. However, the totality of all implications between $\bigvee \bigwedge cs$ or pole-groups must be generated by iterative application of operations according to Wittgenstein's standards (cf. p. 78). This amounts to defining a logical *calculus* for predicate logic satisfying Wittgenstein's standards.[39] Thus, the following problem arises:

problem of implication: The problem of implication is defining the totality of

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[39] cf. Wittgenstein's addition to Ramsey's copy of the *Tractatus* next to 6.02: "The fundamental idea of math. is the idea of *calculus* represented here by the idea of *operation*" and "The beginning of logic presupposes *calculation* ..." printed in Lewy (1967), p. 421 and Nedo (1983), p. 193.

> implications between $\bigvee \bigwedge cs$ (pole-groups respectively) by iteratively applying derivation rules in terms of logical operations.

We will solve this problem in the realm of elementary predicate logic by defining a calculus of logical operations (cf. section 5.3.7). Furthermore, we will generalize this solution to the whole realm of predicate logic (cf. section 6.4). Thus, we will reduce internal relations between predicate formulae to relations of implication between $\bigvee \bigwedge cs$, which are definable by operations.

3.2 Axiomatic vs. iconic proof-conception

In MN, p. 109[5] Wittgenstein describes the "procedure of the old Logic" as follows:

> This is the actual procedure of [the] old Logic: it gives so-called primitive propositions; so-called rules of deduction; and then says that what you get by applying the rules to the propositions is a logical proposition that you have proved.

Wittgenstein refers to logical proofs in terms of derivations within an axiomatic system. Frege's and Russell's systems satisfy this proof conception as well as modern sequence calculi do: A formula is proven by deducing it from the axioms applying derivation rules. Wittgenstein does not deny that logical true formulae or tautologies can be identified by this procedure. However, he emphasizes that their logical truth cannot be proven this way. He goes on to say:

> The truth is, it tells you something about the kind of proposition you have got, viz that it can be derived from the first symbols by these rules of combination [...].

What is proven by the axiomatic proof procedure is simply the derivability of theorems from the axioms. The logical truth of the theorems, however, is not proven as it is based on the logical truth of the axioms. This is not denied within the framework of classical logic. Not Wittgenstein's comment that proofs within an axiomatic system are in need of a meta-logical justification is significant. Instead, it is the fact that his conception of New Logic opposes this common understanding of logical proofs that is remarkable. Throughout his life, Wittgenstein was opposed to understanding logical and mathematical proofs whose existence rested on axioms because one has to rely on some meta-logical, intuitive evidence if one wants to not only maintain the derivability of theorems, but their logical or mathematical correctness. In PG, p. 297 (cf. TLP 6.1271) he says:

> Logic and mathematics are not *based on* axioms, [...]. The idea that they are involves the error of treating the intuitiveness, the self-evidence, of the fundamental propositions as a criterion for correctness in logic.

Axiomatic proofs do not deliver a purely syntactical criterion for logical properties of arbitrary formulae of a formal system. The axioms are taken for granted without a formal proof. They hold an exceptional position within the system, but this position is not justified syntactically – the axioms are formulae within the system and do not differ essentially from other formulae. This can be seen from the fact that there are several correct and complete axiom systems for the same formal system. It can also be seen from the fact that not all axioms share some syntactical feature that identifies them as axioms. The common understanding of logical proofs, in terms of derivations from axioms, depends on proofs of the logical truth of the axioms and of the correctness and completeness of a calculus relative to some previously defined semantics. Such proofs cannot be carried out within formal logic. Thus, the question arises regarding the meta-logical justification of an axiomatic calculus. Such a foundation necessarily exceeds the limitations of admissible evidence in logic. One objective of Wittgenstein's New Logic is to replace axiomatic proof procedures by a proof procedure that is not in need of such a meta-logical foundation. In TLP 6.1265f, he says:

> It is always possible to construct logic in such a way that every proposition is its own proof.
>
> All the propositions of logic are of equal status: it is not the case that some of them are essentially primitive propositions and others essentially derived propositions.
>
> Every tautology itself shows that it is a tautology.

That logical propositions are "their own proof" or tautologies "show themselves" to be tautologies does not mean that there is no need for proofs in terms of syntactical manipulations of formulae in order to identify tautologies as tautologies. It only means that this can be done by relying solely on the formulae themselves as starting points of the proof instead of relying on axioms. In this respect Wittgenstein is looking for something similar to tableaux procedures such as Beth's or Smullyan's procedure (cf. Beth (1962), Smullyan (1995)). Contrary to these procedures, New Logic does not only aim for a procedure to identify tautologies, but for a procedure to identify the truth conditions of "every proposition", i.e. any predicate formula. In Wittgenstein's conception, syntactical proofs in terms of deriving theorems from axioms are replaced by syntactical proofs in terms of converting formulae to symbols of an ideal notation. The purpose of the expressions of this ideal notation is not only to identify tautologies but to identify the truth conditions of *any* formula by their syntactical features. Again and again, Wittgenstein stresses that one has to identify tautologies "from the symbol alone" (TLP 6.113), or that one can "[recognize] in a suitable notation [...] the formal properties of propositions by mere inspection of the propositions themselves" (TLP 6.122). Axioms, i.e. formulae with an exceptional position within a logical system, are not needed in this conception because every formula "is its own proof," assuming a sufficient notation that identifies the truth conditions of all formulae (TLP 6.1265, cf. 6.127f.). A proof does not consist of a derivation of formulae from formulae of the same system, but in the conversion of the formula to the symbols of an ideal notation, according to a general procedure, wholly depending on the syntax of the initial formula. Put concisely, the proof conceptions can be compared as follows.

Proof conception of Old Logic:

$$\text{Axioms} \Rightarrow \text{theorems}$$

The formula in question marks the end of the proof. It has to be a theorem in order to be provable. Proofs of the truth conditions of formulae not being theorems are not available in this conception. This also applies to calculi of natural deduction that do not start their proofs by fixed axioms but prove argument schemata. Not to reveal the truth conditions of the formulae by syntactical features of ideal symbols is the objective of natural deduction proofs but to deduce a formula from certain assumptions.

Proof conception of New Logic:

$$\text{Formula} \Rightarrow \text{ideal symbol}$$

The ideal symbol identifies the truth conditions of the initial formula. In contrast to predicate formulae, its syntactic properties identify its logical properties, i.e. its truth conditions and its internal relations to other formulae. The all important difference is that syntactic properties, i.e. properties recognizable in the symbols themselves, serve as identity criteria of logical properties. In this sense, a "syntactical foundation of logic" means to reduce predicate formulae to ideal symbols that allow one to identify the logical properties in question. It does not mean to define some axiomatic system that is correct and complete according to some standard interpretation.

Wittgenstein exemplifies his proof conception in TLP 6.1203 for propositional formulae by introducing a notation that is similar to the ab-notation. One might also think of the truth-table method as a well known procedure that realizes essential features of this proof conception. For truth-tables, the ideal symbol consists of the columns of "T"s and "F"s below the main sentential connective and the propositional variables in the left part of the truth-table. The construction of the remaining columns are part of a mechanical procedure to construct the column below the main sentential connective (cf. Lampert (2005a), section 6.3, p. 183-188). Consider, for example, the following truth table:

P	**Q**	$\neg$	(	$\neg$	P	$\vee$	$\neg$	Q	)
T	**T**	**T**		F		F	F		
T	**F**	**F**		F		T	T		
F	**T**	**F**		T		T	F		
F	**F**	**F**		T		T	T		

The symbolic features of the table represent the truth conditions of the propositional formula. This can be seen from the fact that the following paraphrase of the truth conditions of the initial formula follows from it:

- $\neg(\neg P \vee \neg Q)$ is true iff P is true and Q is true; and
- $\neg(\neg P \vee \neg Q)$ is false iff
  - P is true and Q is false, or
  - P is false and Q is true, or
  - P is false and Q is false.

The same explanation is generated, for example, in case of $P \wedge Q$ or other equivalent formulae. The objective of the ab-notation is to realize such a proof conception for predicate logic.

By exploring Wittgenstein's New Logic, it shall be demonstrated through purely logical means that understanding logic in terms of an axiomatic theory is superfluous. It should be noted that it is not maintained that axiomatic proof systems are mistaken. However, in logic their form is misleading in that it suggests that logic rests on some truth beyond symbols and their rule-governed manipulation. The form of axiomatic systems evokes problems at the very foundation of the axioms or in the correctness and completeness of the axiomatic system. According to Wittgenstein's point of view, these problems should be solved by changing the logical point of view rather than going beyond it. Thus, with the conception of New Logic, a certain philosophical point of view concerning the understanding and foundation of logic is at stake. The ambitious objective is to stringently justify a Wittgensteinian understanding of logic by constructing a logic of an alternative form without delivering different logical results. In other words, to construct a logic that identifies in a different way the truth conditions of formulae that have been established according to classical semantics.

In Wittgenstein's proof conception syntax and semantic do not fall apart as in classical logic. Through the proof procedure, the truth conditions of the formulae become obvious. Its rules explain what the single syntactical features contribute to express certain truth conditions. In this respect, Wittgenstein's proof conception provides semantics in terms of a theory *defining* truth conditions of formulae. Thus, it does not need to be justified by some prior, independent given semantics. This, of course, does not mean that it cannot be compared to classical semantics. It should be demonstrable that both concepts of semantics are compatible and attribute the same truth conditions to the formulae. However, the truth conditions need not be identified by a procedure external to the syntactical manipulations of the proof procedure itself. On the contrary, every step in the procedure of New Logic is a step in clarifying the truth conditions, and nothing more can or should be done in addition to define the steps explicitly. There is no further story to tell, except to explain the rules of the procedure. Consequently, the question of the correspondence between syntax and semantics is not the focus of Wittgenstein's conception. Rather, the questions focus on how an ideal notation that unambiguously identifies truth conditions of the formulae should look, and how a procedure can be defined in order to convert formulae into the symbols of such an ideal notation.

Wittgenstein's conception differs significantly from the traditional point of view by regarding the syntax of predicate logic as deficient because the truth conditions of predicate formulae cannot be identified by relying on their syntactical features. Repeatedly, he identifies his reason for rejecting the syntax of predicate logic – the "old notation" – as the fact that syntactically different formulae might be equivalent. For example, the first sentence of NL, p. 93[1], is as follows:[40]

> One reason for thinking the old notation wrong is that it is very unlikely that from every proposition p an infinite number of other propositions not-not-p, not-not-not-not-p, etc. should follow.

And in NL, p. 102[3] it says:

> If p = *not–not–p* etc.; this shows that the traditional method of symbolism is wrong, since it allows a plurality of symbols with the same sense; and thence it follows that, in analyzing such propositions, we must not be guided by Russell's method of symbolizing.

Commonly, the language of predicate logic is regarded as an ideal language, in contrast to natural language, for two main reasons: it is set up recursively, and it is unambiguous in so far as every formula expresses a certain truth function of atomic propositions. However, according to Wittgenstein's point of view, this is not sufficient because identical truth functions can still be expressed differently. In this sense, the syntax of predicate logic shares a deficiency compared to natural language. The problem is not primarily that signs of different types are equivalent, but that no general syntactical criterion, in terms of some syntactical feature shared by equivalent formulae, exists to identify equivalent symbols as equivalent (cf. NL, p. 94[3], p. 99[2], p. 101[7]). This is seen by considering equivalent formulae that differ in several respects, such as the following formulae:

$$\exists x_1 \forall x_2 ((Q \land \forall x((\exists y \exists z Ixyz \land \neg Q) \lor (\forall x_3 \exists x_4 Hx_3x_4 \land \neg Q))) \lor ((\neg Fx_2 \land Gx_1) \lor Hx_2x_1)) \tag{3.1}$$

$$\neg \forall y \exists x \neg ((\neg Fx \land Gy \land P) \lor (\neg Fx \land Gy \land \neg P) \lor Hxy) \tag{3.2}$$

$$\exists y \forall x Hxy \lor \exists y (\forall x (\neg Fx \lor Hxy) \land Gy) \tag{3.3}$$

According to classical logic, it is possible to prove their equivalence by deducing one from the other. However, it is not possible to identify a syntactical feature that (3.1) to (3.3) have in common that justifies their equivalence. The fact that the truth conditions cannot be identified by means of the syntax of predicate formulae also becomes evident if one considers non-equivalent formulae; the differences

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[40] Cf. TLP 5.43: "Even at first sight it seems scarcely credible that there should follow from one fact p infinitely many *others*, namely $\sim\sim p$, $\sim\sim\sim\sim p$, etc. And it is no less remarkable that the infinite number of propositions of logic (mathematics) follow from half a dozen 'primitive propositions'."

of their truth conditions cannot be identified by syntactic criteria. Moreover, it cannot even be proven syntactically that the formulae are not equivalent.

In the framework of Wittgenstein's New Logic, laying down axiomatic calculi with certain meta-logical properties is not the first task of logic; the first step is solving the equivalence problem.

equivalence problem: The equivalence problem requires defining a general procedure such that the same symbol is assigned to every predicate formula of a class of equivalent formulae, and different symbols are assigned to non-equivalent predicate formula.

To solve this problem, syntactical differences of equivalent formulae must be minimized systematically. To solve this problem, first for elementary predicate logic and then for the whole realm of predicate logic, is the main task of this book. The merit of Wittgenstein's conception has to be measured against this objective. The decision problem is the special case of the equivalence problem concerning the identification of tautologies or theorems. Thus, its solution is implied in the solution of the equivalence problem.

The symbols assigned to the formulae by an ideal notation – in the case of the ab-notation the "ab-symbols" – shall identify the truth conditions of predicate formulae. This means that the ab-symbols can be paraphrased by a mechanical procedure such that they denote common features of the models and counter-models of the initial formula. This implies the possibility of generating the totality of models and counter-models from the ab-symbol of a formula, without taking into account interpretations individually. This conception will be made precise by elaborating an alternative to classical semantics on the basis of the ab-notation (cf. section 6.3). We call this alternative "New Semantics". Furthermore, we will show how relations of implication between ab-symbols are generated by mere iterative application of operations (cf. section 5.3.3 and 6.4). The problem to generate the totality of models and counter-models of a *wff* by iteration we label "'problem of semantics", whereas the problem to generate the totality of relations of implications between ab-symbols is part of the "problem of implication". We will explain these problems later on in detail (cf. sections 3.1.6 and 3.1.7). At this stage, it suffices to characterize the understanding of logical proofs in the framework of New Logic in general terms, in order to evaluate its elaboration against the realization of this proof conception.

logical proof: A proof according to the conception of New Logic is a syntactical, rule-governed transformation from *wff* to ab-symbols that identifies

1. the conditions of truth and falsehood of *wff*s unambiguously by converting them to their ab-symbols, and
2. internal relations between *wff*s by iteratively generating ab-symbols from ab-symbols.

Thus, a logical proof involves the application of the general procedure to solve the equivalence problem. The purpose of such a proof conception is not to come to knowledge of some truth, but rather to reveal an objective, syntactical criterion in order to identify formal properties and relations of *wff*. The distance this proof-conception will be realized in logic is, in turn, measured against the extent of the solution of the equivalence problem as well as the solutions of the problems of semantics and implication. The complete realization of this proof-conception is the core problem of logic according to New Logic.

We label Wittgenstein's method "iconic proof conception" (as opposed to axiomatic proof conception), as it aims to replace expressions that do not represent their truth conditions with expressions depicting their truth conditions one-to-one by their syntactical features. We hereby make use of Peirce's notion of "icon" and "iconic logic" (cf. Shin (2002)). However, it should be noted that Peirce distinguishes "symbols" that do not resemble what they represent from "icons". As we will see, Wittgenstein's terminology differs significantly in this respect as he uses the term "symbol" and "symbolizing property" in order to identify those features of a sign that do resemble what is represented by that sign.

In fact, no satisfactory answer to the question of the truth conditions of predicate formulae can be put forward in the framework of classical logic. Paraphrases of the formulae explain their truth conditions as little as the formulae themselves. Derivations are only capable of identifying internal relations between formulae. In the framework of classical semantics, no general descriptions of all models and counter-models of a given *predicate* formula can be delivered. Not the infinite totality of models but only single models or counter-models of a given *wff* can be specified (cf. Lampert (2006b) and section 6.3.2). To the issue of the truth conditions of a predicate formula being neither a propositional formula nor a formula of monadic predicate logic, no answer is given in terms of a mechanically produced, finite expression that explicates the truth conditions of the formula in a satisfying manner. This is not only deficient from the point of view of New Logic, but also from the perspective of everyone handling predicate formulae and seeking to understand them.

Though Wittgenstein's New Logic differs in both its proof conception and its identification of relevant problems from classical logic, these two conceptions are not incommensurable. First, by the methods of truth-tables and Venn-Diagrams, procedures exist that can be interpreted as transformations of formulae of subclasses of predicate logic to expressions identifying their truth conditions (cf. also Landini (2007), p. 118-124). Although these procedures do not satisfy the proof-conception of New Logic entirely, they allow one to understand what to look for in the conception of New Logic, even without being acquainted with the ab-notation. Furthermore, the definition of well-formed predicate formulae, as well as their interpretation, is also presumed by New Logic. The conditions of truth and falsehood of a predicate formula are not in question; rather, the question is how to identify them. From a traditional point of view, the objectives of New Logic should not be challenged, only their ability to be realized is questioned. However, instead of putting aside a conception that seems unrealizable according to the traditional point of view, it seems more instructive to judge the extent of what is possible, either by endeavoring to realize such a conception, or by discussing a serious attempt to realize the ideals of New Logic. This work aims to make this possible.

Practical Relevance

The realization of Wittgenstein's New Logic has practical relevance apart from the theoretical relevance for logic, specifically its philosophy and its meta-logic. As will be explained in section 5.4.1.1, it constitutes an analogy to the optimization of propositional disjunctive normal forms in terms of their minimization in predicate logic. The minimization of disjunctive normal forms in propositional logic – e.g. according to the Quine-McCluskey algorithm – is crucial for reducing complexity and for causal reasoning. However, information can often only be represented adequately within predicate logic. Thus, elaborating a manageable concept of disjunctive normal forms in predicate logic, as well as a procedure to minimize them, is significant. As will be shown, the expansion of Wittgenstein's New Logic in the realm of predicate logic makes such a procedure possible.

Furthermore, defining a procedure that explains the truth conditions of predicate formula in a finite number of steps is significant because it provides the only possibility of understanding the formulae in a methodical manner. It removes an often bewailed deficiency that is confronted when explaining and applying non-monadic predicate formulae to formalize natural language (cf. Lampert (2006a)).

Chapter 4

Wittgenstein's Heritage

This chapter identifies essential elements of the ab-notation on the basis of Wittgenstein's remarks. This will serve as a foundation for a systematic elaboration of the ab-notation in Part II. In contrast, this chapter is not a systematic exploration, but rather it provides an understanding of the principles of Wittgenstein's ab-notation.

Wittgenstein's remarks on the ab-notation are primarily concerned with its application in propositional logic. That is why the two following sections, 4.1 and 4.2, are restricted to the ab-notation of propositional logic. Section 4.3 then addresses the issue of the ab-notation in the realm of predicate logic.

Wittgenstein draws a significant distinction between the type of a certain ab-diagram and its symbolizing properties. This distinction will be elucidated in the two following sections.

4.1 ab-diagrams

The rules for converting propositional formulae to ab-diagrams are plain.[1] The formulae are converted by parsing them from the inside to the outside according to their logical hierarchy. First, all occurrences of propositional variables are provided with an a-pole to the left and a b-pole to the right. Thus, instead of p, one has to write apb (cf. CL, letter 28 and 32). Sometimes Wittgenstein attaches the

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[1] The most detailed explanation of the diagrams can be found at TLP 6.1203. However, at this point, Wittgenstein does not use a- and b-poles, but T- and F-poles. In CL, NL, and MN, three diagrams are represented: CL, letter 32, p. 57 (cf. p. 123); NL, B25, printed in Biggs (1996), p. 30 (p. 125); MN, p. 115 (cf. p. 132). The construction of these diagrams is plain if one considers Wittgenstein's remarks on the ab-notation in detail. All these diagrams will be discussed in this chapter.

poles with a hyphen: $a - p - b$ (e.g. NL, p. 94[6], p. 106[3], MN, p. 114f., CL, letter 28). Once, he does not attach the poles to the left and right, but to the left hand side with one upon the other: ${}^{a}_{b}p$ (NL, p. 102[3]). However, he mentions explicitly that these differences in notation are insignificant (CL, letter 28, point (2), p. 47).

Sentential connectives, such as $\neg$, $\wedge$, $\vee$, $\rightarrow$, are translated by ab-operations, i.e. operations assigning a- and b-poles to a- and b-poles in turn. Thus, the formula $\neg p$ is converted to an ab-diagram by first writing down apb and then applying the ab-operation that translates the negator. This operation assigns the b-pole to the a-pole and the a-pole to the b-pole. Thus, one receives $bapba$ as an ab-diagram of $\neg\neg p$ or, alternatively, by making use of hyphens, $b - a - p - b - a$.

Figure 4.1 maps the most common sentential connectives to ab-operations. In any case, the result of applying the ab-operations will again comprise the a- and the b-pole. Negation assigns the a-pole to the b-pole and the b-pole to the a-pole. In addition, there are 14 ab-operations that assign a- and b-poles to the four pairs of poles aa, ab, ba, and bb. Each dyadic sentential connective is represented by one ab-operation. Ab-operations that assign only the a-pole or only the b-pole to all four pairs of poles do not exist, cf. p. 136 for further explanation of this fact.

	$\neg$
a	b
b	a

	$\vee$
aa	a
ab	a
ba	a
bb	b

	$\wedge$
aa	a
ab	b
ba	b
bb	b

	$\vert$
aa	a
ab	a
ba	a
bb	b

	$\rightarrow$
aa	a
ab	b
ba	a
bb	a

	$\leftarrow$
aa	a
ab	a
ba	b
bb	a

	$\leftrightarrow$
aa	a
ab	b
ba	b
bb	a

Figure 4.1: Translation of the most common sentential connectives $\neg, \vee, \wedge, |, \rightarrow$, $\leftrightarrow$ ($\equiv$ respectively) to ab-operations

The relation between Wittgenstein's interpretation of sentential connectives as ab-operations and his criticism of the symbolic representation of the sentential connectives as function symbols in Old Logic will be described below on p. 136.

In the ab-diagrams, a- or b-poles are grouped in pairs using curly brackets (cf. diagram 4.2).[2] Because every dyadic ab-operation assigns both the a- and the b-pole to pairs of poles, the four pairs of poles, aa, ab, ba, and bb, are obtained as bases of further applications of dyadic ab-operations.

For example, the ab-diagram of the formula $p \equiv p$ is generated by first writing down apb twice, then combining the four possible pairs of poles aa, ab, ba, bb by curly brackets. Finally, one applies the definition of the sentential connective $\equiv$ as the operation that assigns the a-pole to aa and bb and the b-pole to ab and ba (cf. figure 4.2).

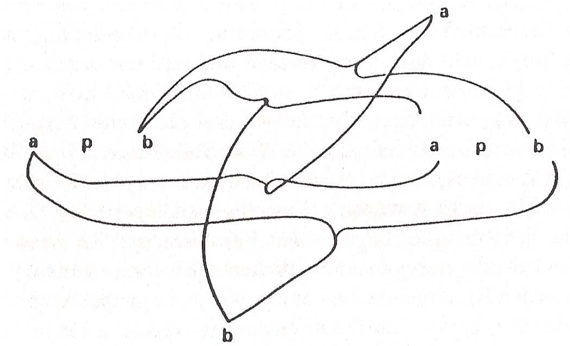

Figure 4.2: Wittgenstein's ab-diagramm of $p \equiv p$ from letter 32 to Russell (CL, p. 57)

This notation contains three crucial differences to truth-tables:

1. Instead of “T” and “F” “a” and “b” are used to denote poles.

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[2] Landini (2007), p. 114f suggests to use Gardner's “shuttles” instead of curly brackets, cf. Gardner (1982), chapter 3. This indeed improves the transparency of ab-diagrams if one adopts them in the way Landini suggests. However, as there is no substantial difference between Landini's use of shuttles and curly brackets and as we will basically refer to the pole-group notation later on (cf. p. 128 and p. 246), we abstain from using shuttles. Furthermore, using shuttles might also be confusing as Gardner's use of them implies several differences to the ab-notation: (i) he refers to only one pole, (ii) he does not represent complex formulae by only one shuttle diagram but splits them up into several ones, (iii) he uses two different sorts of shuttles (straight and dotted lines) and also introduces crosses and half-crosses, and (iv) he rather considers the conclusiveness of argument schemata than looks for a uniform representation of the truth conditions of formulae.

2. In truth-tables, 2^n combinations of truth values are taken into consideration, depending on the n propositional variables of different types that occur in the propositional formula. However, in the ab-diagrams, either two poles (in the case of negation) or four possible combinations of poles (in all other cases) are taken into consideration, regardless of whether or not the propositional variables are identical.

3. In truth-tables, it is impossible to combine opposed truth values (T and F) of the same propositional variable, whereas in the ab-diagrams it is possible to combine opposed poles (a and b) of two occurrences of the same propositional variable.

Point (1) will be discussed below, cf. p. 132. Points (2) and (3) result from the fact that the construction of ab-diagrams relies on the *occurrences* of propositional variables, whereas the construction of truth-tables relies on the *type* of propositional variables. This allows for a combination of poles with two occurrences of the same propositional variable in an ab-diagram. From this it follows that tautologies are not identified by the non-existence of an outmost b-pole, whereas in truth tables, tautologies are identified by the fact that the truth value F does not occur below the main sentential connective. Instead, in reference to ab-diagrams, tautologies are identified by the fact that the outmost b-pole is assigned to at least one pair of opposed innermost poles connected to two occurrences of the same propositional variable in any case, cf. CL, letter 30, p. 53. Wittgenstein applies this general rule to the above ab-diagram of the formula $p \equiv p$, CL, letter 32, p. 60:

> [...] it is tautological because b is connected only with those pairs of poles that consist of opposite poles of a single proposition (namely p).

Thus, identifying tautologies is traced back to the relation of the outmost b-pole to the innermost poles. “a complex pole of propositional logic” is composed of a propositional variable and one innermost pole, e.g. $a - p$ or $b - p$. Then, in general, identifying truth conditions of the formulae is traced back to the relation of the outmost poles to classes of complex poles. Opposed complex poles of propositional logic are complex poles with identical propositional variables, but opposed poles. For example, $a - p$ and $b - p$ are opposed complex poles, whereas $a - p$ and $b - q$ are not opposed. The fact that the outmost pole of an ab-diagram might be connected to opposed complex poles marks a difference between the ab-diagrams and truth-tables. As will be shown below, this point is crucial for applying the ab-notation to predicate logic (cf. p. 150). Ignoring this difference, or even maintaining that the rules of the ab-notation should be adjusted to the method of truth-tables in this respect as suggested by Black (1964), p. 323f, makes it impossible to adequately understand the ab-notation.

In the 1979 edition of NL that is based on the "Costello Version," a diagram from a supplementary sheet of Russell's original manuscript RA823 of NL, namely folio 25, figure 4.3, taken from Biggs (1996), p. 30, is not printed.[3]

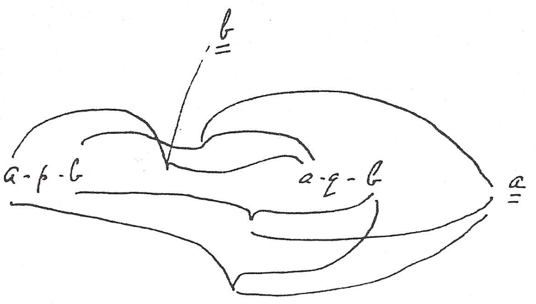

Figure 4.3: ab-diagram of $p \mid q$ in RA823, f.25

This diagram induced misunderstandings. Biggs considers it to be added by Russell later on, Biggs (1996), p. 22-25. However, this cannot be the case, as Wittgenstein refers in CL, letter 28, point (8), p. 48 to this diagram. He writes:

> (8) The exact ab-indefinable is given in the manuscript.

Wittgenstein refers to a German manuscript, which is most likely a German dictation of Wittgenstein taken at Berlitz school in the beginning of October 1913. This then served as draft for Russell's translations of NL, MS 1-4 (cf. Geschkowski (2000), p. 14). One can see this from the German citation in question (7) and from CL, letter 27 from 29.10.1913, where Wittgenstein refers to "a copy of my manuscript". Wittgenstein's answer concerns a question asked by Russell, pertaining to NL, p. 102, last paragraph. Question (7) refers to a passage directly preceding this paragraph, namely NL, p. 102[6], cf. CL, letter 29, question (7), p. 50. Question (9) refers to a later passage, namely NL, p. 105[9], cf.

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[3] TS$_x$ of NL, f.14 has the same diagram. The *Notes on Logic* were written down by Russell and rely on dictations and German manuscripts of Wittgenstein. The history of NL is described by McGuinness (1972), Biggs (1996) and Geschkowski (2000), p. 9-15.

McGuinness (1972), p. 447. The last paragraph of NL, p. 102 is between these two passages. Thus, question (8) refers to a passage between the two passages that questions (7) and (9) refer to. In the last paragraph of NL, p. 102, Wittgenstein mentions $p \mid q$:

> [. . .] this expression says something indefinable about all arguments p and q. [. . .] $p \mid q$ is merely a mechanical instrument for constructing all possible *symbols* of ab-functions.

$\mid$ is the Sheffer stroke. As is well known, it is ambiguous which operation the Sheffer stroke denotes. Either $\mid$ means "neither . . . nor . . ." or "not both: . . . and . . .". According to the former interpretation $p \mid q$ is equivalent to $\neg p \wedge \neg q$, according to the latter to $\neg p \vee \neg q$. In NL, p. 102[6], Wittgenstein uses the Sheffer operation explicitly in the latter sense. Thus, Wittgenstein refers to a diagram in the German manuscript, presumably on a supplementary sheet, that served as draft for Russell's translations. The cited diagram, RA 823, f.25, is a copy of the diagram to which Wittgenstein refers. It represents the formula $p \mid q$ and makes use of the Sheffer stroke in the sense of "not both: . . . and . . .", or, more precisely, in the sense of an operation assigning the a-pole to ab, ba, and bb, and the b-pole to aa, cf. the definition of the Sheffer stroke in figure 4.1.

However, Russell's additional remark "This is the symbol for $\sim p \vee \ \sim q$" is misleading. "$\sim$" is Russell's symbol of negation, $\sim p \vee \ \sim q$, the indefinable "primitive idea" of propositional logic in PM, cf. p. 139. Russell confuses the indefinable primitive idea of PM with the Sheffer-operation. This is misleading because the ab-diagram of $\sim p \vee \sim q$ is as follows:

It is correct to say that the different ab-diagrams of figure 4.3 and 4.4 represent equivalent formulae. They symbolize the same truth conditions. As the following section will show, it is also appropriate to say that both formulae are symbolized by the same indefinable *ab-symbol*, despite the fact that they are represented by different *ab-diagrams*. However, in contrast to $\sim p \vee \sim q$ the formula $p \mid q$ does not contain any negation sign. This difference in the formulae is represented by a difference in the ab-diagrams. Only by noting the difference in symbolizing and non-symbolizing properties of ab-diagrams does it become plausible that all "*symbols* of ab-functions" or truth function can be constructed by applying the Sheffer operation, cf. the cited passage from NL, p. 102 above. This will become clear in the following section.

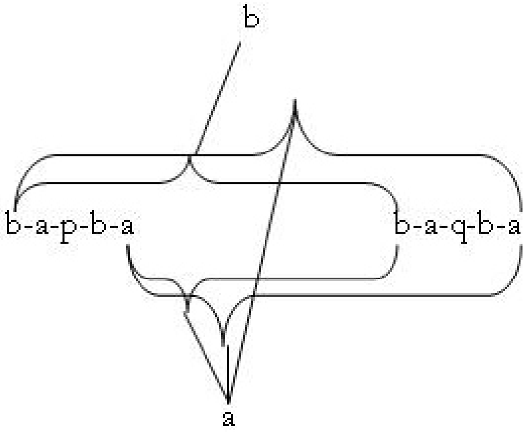

Figure 4.4: ab-diagram of $\sim p \vee \sim q$

4.2 ab-symbols

Ab-diagrams do not solve the equivalence problem because different ab-diagrams are assigned to equivalent, but distinct, formulae. However, not all of the syntactic differences between ab-diagrams are different symbolizing properties. The distinction between symbolizing and non-symbolizing properties is fundamental to adequately interpreting the ab-notation, cf. NL, p. 99[2]:[4]

> In regard to notation, it is important to note that not every feature of a symbol symbolizes.

Symbolizing properties are those syntactic properties of the ab-diagrams that are significant in identifying the conditions of truth and falsehood of the initial formula. Wittgenstein mentions repeatedly that, in the ab-diagrams, and analogously in truth-tables, cf. NL, p. 94[3], only the connection of the outmost poles to the inmost poles is a symbolizing property, whereas all poles in between are irrelevant for the representation of the conditions of truth and falsehood of the initial formula. He also expresses this by saying that the relation of the outer poles

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[4] Wittgenstein uses the verb "to symbolize" as a technical term in NL and MN. In contrast to its common usage, he uses this verb as an intransitive verb. Thus, he also makes use of the termini "symbolizing property / feature" and "symbolizing fact". This usage stems from the distinction of syntactic features that are needed to identify the truth conditions represented by an ab-diagram, and others which are not significant for identifying truth conditions. We adopt Wittgenstein's use of the verb to "symbolize" in this book.

to the inner poles is transitive, NL, p. 102[4], MN, p. 114[7], CL, letter 28, p. 48, point (4). Thus, in $a - b - a - b - a - p - b - a - b - a - b$ the outmost poles have the same relation to the inmost poles as the outmost poles have to the inmost poles in $a - b - a - p - b - a - b$. Despite the fact that the concatenation of poles differs, the relation of the outmost to the inmost poles is the same in each case. Therefore, the symbolizing property is the same, though the types of the two ab-diagrams differ.

Wittgenstein distinguishes the identity criteria of ab-diagrams and ab-symbols: ab-diagrams differ if they differ in type. On the other hand, ab-diagrams are different ab-symbols only if their symbolizing properties differ. He illustrates this distinction by the simplest example: the ab-diagram of p, namely $a - p - b$, and the ab-diagram of $\neg\neg p$, namely $a - b - a - p - b - a - b$. Although the two ab-diagrams differ, their symbolizing properties are identical because the outmost a-pole is connected with the inmost a-pole[5], and the outmost b-pole with the inmost b-pole in both cases. Thus, the two equivalent formulae, p and $\neg\neg p$, are represented by different ab-diagrams in the ab-notation, but by the same ab-symbol.

To represent ab-symbols without reproducing insignificant syntactic properties of ab-diagrams, Wittgenstein describes a simplification of ab-diagrams resulting in single “pole-groups”, NL, p. 102[4], or “classes of poles”, CL, letter 30:

> In place of every proposition „p“, let us write „${}^{\mathrm{b}}_{\mathrm{a}}\mathrm{p}$“. Let every correlation of propositions to each other [. . .] be effected by a correlation of their poles „a“ and „b“. Let this correlation be transitive. Then accordingly „${}^{\mathrm{b-b}}_{\mathrm{a-a}}\mathrm{p}$“ is the same symbol as „${}^{\mathrm{b}}_{\mathrm{a}}\mathrm{p}$“. Let n propositions be given. I then call a „class of poles“ of these propositions every class of n members, of which each is a pole of one of the n propositions, so that one member corresponds to each proposition. I then correlate with each class of poles one of two poles (a and b). The sense of the symbolizing fact thus constructed I cannot define, but I know it.

The simplification of ab-diagrams does not contain any intermediary poles, only innermost poles connected to propositional variables and outmost poles connected to the respective pole-groups. We will call the notation assigning outmost poles to classes of poles, or pole-groups, the “pole-group notation”. Due to the simplicity of the pole-group notation, the propositional variables shall be provided with poles on their left hand side in each case. Furthermore, poles shall be

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[5] For $a - p - b$, the outmost and inmost poles are identical. In accordance with Wittgenstein’s rule of transitivity and NL, p. 102[3] one might replace $a - p - b$ by $a - a - p - b - b$ in order to distinguish outmost and inmost poles in either case, cf. p. 156.

assigned by a hyphen to classes of poles and propositional variables. Thus, the two ab-diagrams, $a - p - b$ and $a - b - a - p - b - a - b$, are represented by the same pole-groups $a - \{a - p\}, b - \{b - p\}$. The same applies to the abovementioned ab-diagrams of the equivalent formulae , $p \mid q$ and $\neg p \ \vee \neg q$, cf. figures 4.3 and 4.4:

$$\begin{aligned} &a - \{a - p, \quad b - q\}, \\ &a - \{b - p, \quad a - q\}, \\ &a - \{b - p, \quad b - q\}, \\ &b - \{a - p, \quad a - q\}. \end{aligned}$$

By converting the ab-diagram of figure 4.2 to pole-groups, Wittgenstein's rule of identifying tautologies can be applied directly:[6]

$$\begin{aligned} &a - \{a - p, \quad a - p\}, \\ &a - \{b - p, \quad b - p\}, \\ &b - \{a - p, \quad b - p\}, \\ &b - \{b - p, \quad a - p\}. \end{aligned}$$

The initial formula is tautologous because all b-pole-groups contain opposite, complex poles.

The ab-notation aims to identify the truth conditions of formulae by converting them to ab-symbols, such that the same ab-symbol is assigned to all equivalent formulae. Wittgenstein's rule of transitivity does not suffice in this case, even in the realm of propositional logic. There are still equivalent propositional formulae that are converted to different pole-groups by merely applying the rule of transitivity. The two formula p and $(p \wedge r) \vee (p \wedge \neg r)$, e.g., are equivalent, but as long as one only applies the rule of transitivity, different pole-groups are obtained from their ab-diagrams:

---

[6] CL, p. 53: „[...] if the b-Pole is connected to such *groups of inside Poles ONLY contain opposite poles of ONE prop[osition]*, then the whole prop[osition] is a true, logical prop[osition].“

$$\begin{array}{ll}
 & a-\{a-p, \quad a-r, \quad a-p, \quad a-r\},\\
 & a-\{a-p, \quad a-r, \quad a-p, \quad b-r\},\\
 & a-\{a-p, \quad a-r, \quad b-p, \quad a-r\},\\
 & a-\{a-p, \quad a-r, \quad b-p, \quad b-r\},\\
 & b-\{a-p, \quad b-r, \quad a-p, \quad a-r\},\\
 & a-\{a-p, \quad b-r, \quad a-p, \quad b-r\},\\
 & b-\{a-p, \quad b-r, \quad b-p, \quad a-r\},\\
a-\{a-p\}, & b-\{a-p, \quad b-r, \quad b-p, \quad b-r\},\\
b-\{b-p\}. & b-\{b-p, \quad a-r, \quad a-p, \quad a-r\},\\
 & a-\{b-p, \quad a-r, \quad a-p, \quad b-r\},\\
 & b-\{b-p, \quad a-r, \quad b-p, \quad a-r\},\\
 & b-\{b-p, \quad a-r, \quad b-p, \quad b-r\},\\
 & b-\{b-p, \quad b-r, \quad a-p, \quad a-r\},\\
 & a-\{b-p, \quad b-r, \quad a-p, \quad b-r\},\\
 & b-\{b-p, \quad b-r, \quad b-p, \quad a-r\},\\
 & b-\{b-p, \quad b-r, \quad b-p, \quad b-r\}.
\end{array}$$

The elements of the pole-groups are complex poles. They are the literals of the ab-notation. In propositional logic, they consist of an a- or a b-pole connected to a propositional variable. Pole-groups are classes of complex poles. “Ab-symbols”, which satisfy the requirements of ideal symbols, are only listings of a- and b-pole-groups that are identical for equivalent formulae, cf. NL, p. 94[3], p. 99[2]. As the example shows, the two complex poles $a-r$ and $b-r$ are superfluous in the pole-groups of formula $(p \wedge r) \vee (p \wedge \neg r)$. Their occurrence is not a symbolizing property in the ab-diagram of $(p \wedge r) \vee (p \wedge \neg r)$. Thus, the list of pole-groups of this formula is not the ab-symbol. The listed pole-groups still contain syntactical features that are superfluous for identifying truth conditions of the initial formula. In order to solve the equivalence problem, Wittgenstein’s rule of transitivity must be complemented by additional rules that allow one to identify the symbolizing properties. However, the available manuscripts do not contain any further suggestions on this topic. Yet, Wittgenstein does not presuppose that his rule of transitivity is sufficient to identify the symbolizing properties of ab-diagrams. Rather, by remarking that equivalent formulae, such as p and $p \wedge p$, $\neg p$ and $p \mid p$ or

$p \wedge q$ and $(p \mid q) \mid (p \mid q)$, cf. NL, p. 103[1], need to be represented by identical ab-symbols, he presupposes that not all occurrences of complex poles make up ab-symbols. The definition of rules that are sufficient to assign identical ab-symbols to equivalent formulae is the main task of the complete elaboration of the ab-notation in Part II of this book. As will be seen, only the maximal number of different ab-pole-groups that cannot be further minimized are ab-symbols.

Wittgenstein calls ab-symbols "symbolizing facts" and deems their sense undefinable, cf. the above quoted passage NL, p. 102[4]. He calls them "symbolizing facts" because the symbolizing properties of the ab-symbols are *structural properties* and not stipulations about the meaning of the single signs. He explains this by referring to the interpretation of the ab-diagram of figure 4.5 in MN, p. 115[4,5]:[7]

> This symbol might be interpreted either as a tautology or a contradiction.
>
> In settling that it is to be interpreted as a tautology and not as a contradiction, I am not assigning a *meaning* to a and b; i.e. saying that they symbolize different things but in the same way. What I am doing is to say that the way in which the a-pole is connected with the whole symbol symbolizes in a *different way* from that in which it would symbolize if the symbol were interpreted as a contradiction. And I add the scratches a and b merely in order to shew in which ways the connection is symbolizing, so that it may be evident that wherever the same scratch occurs in the corresponding place in another symbol, there also the connection is symbolizing in the same way.

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[7] Taken from Biggs (1996), p. 26. Instead of reproducing the diagram of the original, the editors of the first edition of MN reproduced some other diagram, namely the diagram from Wittgenstein's letter to Russell (cf. figure 4.2). Biggs maintains that this original reproduction is accurate, but he is mistaken for several reasons. First, Biggs argues that the diagram in MN would be incorrect because it would not symbolize a tautology as is presumed in the text, cf. Biggs (1996), p. 25-27, following the erroneous argument of Iglesias (1981), p. 318. This is a misjudgement. The outmost b-pole is only connected with opposed complex poles – the only b-pole-group is $b - \{a - p, b - p\}$. Thus, the diagram represents a tautology according to Wittgenstein's rule of identifying tautologies. Biggs erroneously concludes, from the fact that the ab-diagram of figure 4.5 is not identical to the ab-diagram of figure 4.2, that they are not simultaneously capable of representing a tautology. This is a fallacy. One diagram depicts the tautology $p \equiv p$, the other the tautology $p \leftarrow p$. Moreover, Biggs (1996), p. 27 mistakenly states that the diagram in MN should represent the tautology $p \equiv\sim (\sim p)$. Finally, Biggs maintains the diagram is identical to the diagram of NL written by Russell, cf. figure 4.3. This too is mistaken in several respects. Russell's diagram contains the propositional variables p and q and does not represent a tautology. Moore's diagram contains only the propositional variable p and does represent a tautology. Furthermore, they depict different formulae with distinct sentential connectives. Russell's diagram pictures the formula $p \mid q$, while Moore's shows the formula $p \leftarrow p$.

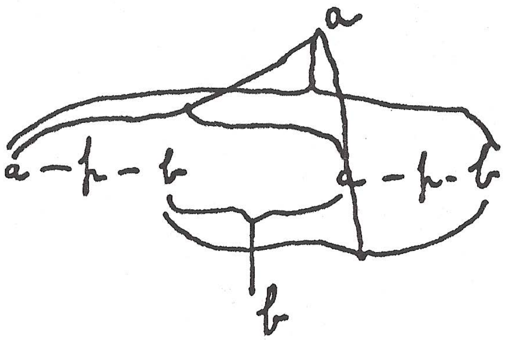

Figure 4.5: ab-diagram of $p \leftarrow p$, original written by Moore, cf. MN, p. 115

Wittgenstein points out that interpreting ab-diagrams does not involve assigning some meaning to the a- or b-pole, for example, by assigning the truth value T to the a-pole and the truth value F to the b-pole. First, he does not want to presume truth values in terms of objects that are denoted by the a- or b-pole, respectively. Instead of assuming *objects* as "True" or " False," he only presumes *conditions* of truth and falsehood. Furthermore, all intermediary poles are insignificant according to the interpretation of ab-diagrams. Thus, it would be false to assign to the tokens "a" and "b" a meaning independent of their position in the ab-diagram. They are the bases of the application of ab-operations and nothing else. Their whole function is to compute the connection of the outmost poles to the inmost poles. To avoid any interpretation of the a- and b-poles in terms of names of truth values Wittgenstein does not use the signs "T" and "F" in the ab-notation. Instead, he only refers to the structural properties in terms of the relationship between particular signs while interpreting ab-diagrams. Wittgenstein also calls structural properties of a logical notation "logical properties". In MN, p. 114[4] he remarks:

> The important thing is that the interpretation of the form of the symbolism must be fixed by giving an interpretation to its *logical properties, not* by giving interpretations to particular scratches.

The shape of the particular characters "a" and "b" does not have meaning as only the connections between the outermost and inner poles are meaningful. Thus, it is the combination of characters that is significant, not the labels of the characters themselves. As such, interpreting the ab-diagrams relies on *facts*, cf.

MN, p. 115[6]. Due to this interpretation of the symbolizing features of the ab-diagrams, ab-symbols constitute "symbolizing facts", cf. also NL, p. 102[4,8]. According to this conception, a reference to truth values is superfluous. Rather, one must interpret ab-symbols as functions that explain the truth and falsehood of the initial formulae, by depending on the truth and falsehood of un-analysable propositions. In propositional logic, these propositions are atomic propositions.[8] Un-analysable propositions are depicted by the complex poles in ab-symbols. For propositional logic, complex poles of the form $a - \xi / b - \xi$ identify the truth / falsehood of atomic propositions: ξ is a meta-variable of propositional variables that, in turn, are variables of atomic propositions.

Ab-symbols represent "ab-functions", cf. NL, p. 94[6], p. 95[4], p. 105[2,3]. Ab-functions express truth functions by assigning the a- and b-pole to classes of complex poles. As a stipulation, ab-diagrams with the outmost b-pole only connected to pole-groups containing opposed complex poles represent *tautologies*. In other words, tautologies are expressions without consistent conditions of falsehood. As a result, the b-pole-groups are interpreted as representing the conditions of falsehood, whereas the a-pole-groups represent the conditions of truth of a formula. Without assigning ab-functions to truth functions, by stipulation, the same ab-diagram can be interpreted as a symbol of opposite truth functions. For example, a given ab-diagram can be interpreted as a symbol of either a tautology or a contradiction.

For Wittgenstein, truth functions are primitive. They are not *defined* by ab-symbols. This follows from the fact that defining what ab-symbols symbolize requires references to truth functions, unless one wants to presuppose truth values as primitive. The ab-diagram of figure 4.5 can be interpreted either as tautology or as contradiction. One defines what the ab-symbol symbolizes by stipulating that *tautologies* are symbolized by the fact that the outmost b-pole is only connected with pole-groups containing opposite inmost complex poles. The purpose of ab-symbols is not to *define* truth functions, but to *unambiguously represent* truth functions of predicate formulae.

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[8] In TLP, Wittgenstein analyses *any* proposition as a function of the truth and falsehood of atomic propositions in terms of propositional logic (TLP 5). This implies his TLP-view that predicate formulae in general express functions of the truth and falsehood of atomic propositions. However, we criticized this view in section 3.1.7 and argue that, in the ab-notation, predicate formulae are analysed as representing functions complex poles in general. For this reason, we use the term "unanalysable proposition" as the more general term than the TPL-term "atomic propositions" as we have neither yet defined the notion of complex poles, nor the equivalent in predicate logic; closed structures.

Truth functions are the “sense” of the ab-symbols that is indefinable, but known, as Wittgenstein is quoted as saying above (cf. NL, p. 102[4]). This means that it is only possible to explain the sense of ab-symbols by paraphrasing them using natural language whose meaning is known. ab-symbols are nothing but a “neater translation” of the “cumbrous” natural language expressions of truth functions, MN, p. 117[4]. This becomes clear when ab-symbols are paraphrased in predicate logic (cf. section 6.3.3). In propositional logic the paraphrases of ab-symbols are straightforward. The ab-symbol of $p \vee q$ is (cf. section 5.4.1):

$$\begin{aligned}&a - \{a - p\},\\ &a - \{a - q\},\\ &b - \{b - p, \qquad b - q\}.\end{aligned}$$

The paraphrase of this ab-symbol in accordance to the initial formula is: “$p \vee q$ is true iff p is true or q is true; $p \vee q$ is false iff p is false and q is false.” The a-pole-groups identify the conditions of truth (= models), while the b-pole-groups identify the conditions of falsehood (= counter-models) of the initial formula.[9] Each singular pole-group identifies a sufficient condition of truth / falsehood; each complex pole of propositional logic identifies a condition in terms of the truth / falsehood of atomic propositions. The paraphrases of the single complex poles are connected by “and;”, the paraphrases of the a-pole-groups and the b-pole-groups, respectively, are connected by “or”.

One must distinguish between ab-functions and ab-operations.[10] According to Wittgenstein, ab-functions are not operations because their arguments are classes of complex poles, and their values are a- or b-poles. The values of ab-functions cannot be their arguments because classes are not identical to poles. In contrast, ab-operations are not functions according to Wittgenstein’s terminology be-

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[9] We will explain the internal relations between pole-groups and models / counter-models in detail in section 6.3. Here, we address the problem of semantics. In particular, we clarify in what sense models and counter-models are *identified* by pole-groups in section 6.3.4. In section 6.3.5 we explain how the models of a *wff* can be generated from its a-pole-groups and the counter-models from its b-pole-groups.

[10] Wittgenstein does not stick strictly to this distinction in NL. On p. 102[8], for example, he uses “ab-functions” in sense of “ab-operations” when he speaks of the “repeated application” of ab-functions. However, at the end of the next paragraph (NL p. 103[1]), he then distinguishes ab-functions and ab-operations by saying that the operation $|$ generates all possible symbols of ab-functions, cf. NL, p. 94[6]. In TLP, Wittgenstein then clearly delineates the difference between operations and functions and, in consequence, between truth operations and truth functions, cf. TLP 4.122-4.128, 5.2-5.5151.

cause their bases and results are always a- or b-poles. They can be applied iteratively. Ab-functions symbolize truth functions, whereas ab-operations do not have a meaning on their own. They are part of the rules needed to construct an ab-diagram of a formula. By applying ab-operations, ab-functions are produced from ab-functions. Or in the terminology of TLP, by applying truth-operations, truth functions are produced from truth functions, cf. TLP 5.3. Different applications of ab-operations can result in the same ab-function. Specifying which structural properties of ab-diagrams symbolize a particular truth function determines which ab-operations are assigned to the particular sentential connectives. For example, let the fact that the outmost b-pole of figure 4.6 is only connected to the inmost complex poles $a - p$ and $b - q$ symbolizes that a formula is false iff p is true and q is false. It then follows that the ab-operation that assigns b only to the pair ab must be assigned to $\rightarrow$, according to the latter's common definition.

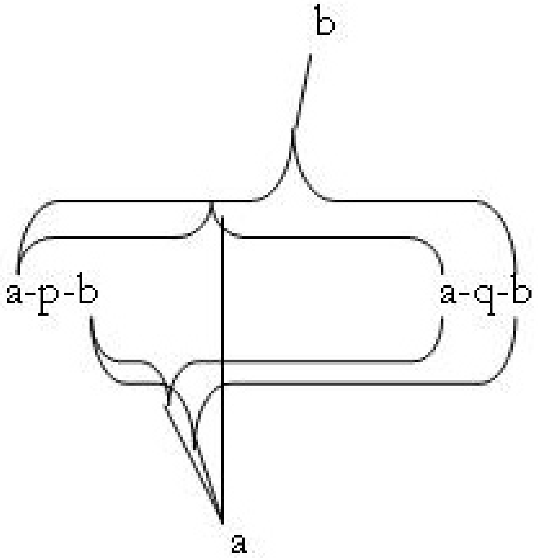

Figure 4.6: ab-diagram interpreted as a symbol of the truth function “if p then q”

Applying the singular ab-operation defining | suffices to generate all ab-functions systematically in propositional logic. Tautologies and contradictions are truth functions, but not ab-operations. It is a misunderstanding to assign them to sentential connectives. There are only 14 dyadic ab-operations depicting sentential connectives, but 16 functions of the truth and falsehood of two atomic propositions. Applying ab-operations leads to ab-diagrams symbolizing tautologies and contradictions. Tautologies and contradictions are not represented by certain ab-operations or sentential connectives. Instead, they result from certain applications of ab-operations or certain concatenations of propositional variables by sentential connectives.

Wittgenstein's distinction between ab-functions and ab-operations is a crucial part of his criticism of the symbolism of the Old Logic. Russell did not understand this distinction, nor Wittgenstein's notion of the symbolizing properties of ab-symbols, cf. CL, letter 29, point (4), p. 50. In a letter to Wittgenstein he writes:

> What you call ab-functions are what the Principia calls 'truth-f[unctio]ns'. I don't see why you shouldn't stick to the name 'truth-f[unctio]ns'.

Wittgenstein, unwilling to deliver rambling explications, and yet not clear about the distinct terminology and exact interpretation of his new notation, rejects Russell's proposal to identify the two terms in question without comment, cf. CL, letter 28, point (3), p. 48. In fact, the concept of "truth functions" in PM ignores the distinction between function and operation that is crucial for Wittgenstein. Truth functions are introduced in PM, p. 8 as follows:

> We may call a function $f(p)$ a "truth-function" when its argument p is a proposition, and the truth value of $f(p)$ depends only upon the truth-value of p.

According to PM, propositions of propositional logic are either atomic or complex. Truth functions are by no means interpreted solely as functions of the truth and falsehood of atomic propositions. Rather, each negation, disjunction, conjunction, or implication is understood as a truth function of the partial expression that the respective sentential connective connects. For example, $\neg\neg p$ is a truth function of $\neg p$, while $\neg(\neg p \vee \neg q)$ is a truth function of $(\neg p \vee \neg q)$. This classical conception of truth functions is rejected by Wittgenstein. Negation, disjunction, conjunction, and implication are truth-operations and not truth functions according to his terminology. The significance of Wittgenstein's distinction is not simply one of terminology; it concerns the main issue of New Logic. Wittgenstein does not object to interpreting formulae as truth functions of their parts, but rather to the symbolic representation of negation, disjunction, conjunction, and implication.

In PM, they are symbolized by sentential connectives, which are used as function symbols. The negator $\neg$ is a monadic function symbol, while $\vee, \wedge, \rightarrow$ are dyadic function symbols. However, according to Wittgenstein negation, disjunction, conjunction, and implication are operations. This understanding is expressed in the ab-notation by representing the sentential connectives by operations that assign poles to poles. Not only are the understanding of negation, disjunction, conjunction, and implication at stake, but also the adequate form of their symbolic representation. Different forms of symbols determine different symbolizing properties of the syntax. According to the conception of New Logic, providing the necessary symbolizing properties to identify truth functions by syntactical features is the criterion of an adequate logical notation. Thus, from the perspective of New Logic, the symbolic alternative of function symbols and operations is not at all an arbitrary convention. Instead, in symbolically representing negation, disjunction, conjunction, and implication with operations, New Logic makes it possible to identify equivalent though syntactically different formulae by means of identical symbolizing properties. This is not possible by relying on the classical "old notation".

Wittgenstein illustrates this once more by the simplest example of the equivalent formulae $\neg\neg p$ and p. According to the conception of Old Logic, $\neg\neg p$ is a truth function of $\neg p$. However, p does not contain $\neg p$ as argument. Therefore, both formulae cannot be identified by their syntactical features as formulae of the same truth function. However, both formulae do express the same truth function of atomic propositions; they are equivalent. Due to the symbolic representation of sentential connectives as function symbols, no syntactical criterion is provided that is common to all equivalent formulae. To the contrary, by representing negation by an ab-operation in the ab-notation, the ab-diagrams of p and $\neg\neg p$ have the same symbolizing properties. The two formulae are represented by the same ab-symbol, namely $a - \{a - p\}, b - \{b - p\}$. This ab-symbol does not contain the ab-symbol of $\neg p$ as $\neg\neg p$ contains $\neg p$, cf. NL, p. 99[2], p. 103[1], TLP 5.44.

Analogously, Wittgenstein argues in case of the possibility that all truth functions may be represented by applying one operation – the Sheffer-operation.[11] Truth functions are by no means *identified* or *defined* by applying the Sheffer operation. It is ab-symbols that identify truth functions. Sheffer's operation, to the contrary, *systematically constructs* all possible truth functions in the realm of propositional logic. By the use of the Sheffer-stroke, for example, p is expressed by $(p \mid p) \mid (p \mid p)$, while $\neg p$ is rendered by $p \mid p$. However, it does not follow

---

[11] Cf. NL, p. 94[6]: "The symbolising fact in a–p–b is that, SAY a is on the left of p and b on the right of p; then the correlation of new poles is to be transitive, so that for instance if a new pole a in whatever way i.e. via whatever poles is correlated to the inside a, the symbol is not changed thereby. It is therefore possible to construct all possible ab functions by performing one ab operation repeatedly, and we can therefore talk of all ab functions as of all those functions which can be obtained by performing this ab operation repeatedly."

that p is a truth function of $\neg p$ in so far as $p \mid p$ is contained in $(p \mid p) \mid (p \mid p)$. The *ab-symbol* of $(p \mid p) \mid (p \mid p)$ is the same as of p, and it does not contain the *ab-symbol* of $\neg p$: "The symbols arising by repeated application of the symbol '|' do *not* contain the symbol '$p \mid q$'." (NL, p. 103[1]).

Contrary to the ab-notation, the distinction between symbolizing and non-symbolizing properties is not available in the classical syntax of propositional logic. This is because it does not distinguish symbolically between the application of operations and the representation of truth functions. This is distinguished in the ab-notation as the identity of truth functions is expressed by the identity of ab-symbols, whereas syntactical differences of equivalent formulae are expressed by differences in the application of ab-operations in the ab-diagrams. As a consequence, ab-notation makes it possible to distinguish symbolizing and non-symbolizing properties. This enables us to identify identical truth functions by their identical symbolizing properties despite their different syntactic representations.

The criticism of the symbolic representation of truth operations by sentential connectives is one example of Wittgenstein's general charge against Old Logic. Old Logic does not differentiate functions and operations symbolically, cf. WVC, p. 216f:

> An empirical totality is traceable back to a *propositional function*; a system to an *operation*. The logical particles are truth-operations. Thus the meaning of the word 'or' is the operation that turns the sense of the propositions 'p', 'q' into the sense of the proposition 'p or q'. [...] An operation is completely different from a function. A function cannot be its own argument. An operation, on the other hand, can be applied to its own result. In mathematics we must always be dealing with systems, and not with totalities. Russell's basic mistake consists in not having recognized the essence of a *system* while representing empirical totalities and systems by means of the same symbol – a propositional function – without drawing any distinctions.

Ab-symbols are the primitive symbols of the ab-notation. They cannot be explicated any further by syntactical transformations. In contrast, predicate formulae are capable of further explanation by converting them to ab-symbols. The crucial difference between predicate formulae and ab-symbols is that it is not possible, in general, to interpret a predicate formula such that its syntactical, structural features identify the truth function it represents. The two equivalent formulae, $p \wedge q$ and $\neg(\neg p \vee \neg q)$, for example, do not share a common structure. As a result, they cannot be identified as representing the same truth conditions by their syntactical properties. This is also, in general, why paraphrasing the structure of predicate formulae do not explicate their truth conditions.

The syntax of the predicate formulae does not provide a way to identify the truth conditions by the syntactical properties of the formulae. This represents a deficiency of the symbolism of Old Logic. It does not allow for identifying the truth function of equivalent, but syntactically different formulae through identical syntactic properties. However, this is precisely the intention of the ab-notation. This distinction is achieved by two means: (i) by replacing the sentential connectives by ab-operations in the ab-diagrams, and (ii) by stipulating rules to interpret the syntactical properties of the ab-diagrams, such that the truth functions of the initial formula can be identified unambiguously by the symbolizing structural properties of the ab-diagrams. Such rules are defined completely if the equivalence problem is solved by converting ab-diagrams to ab-symbols. Defining such rules is the main task of Part II. Before we proceed, Wittgenstein's sketchy remarks on the ab-notation of predicate logic will be addressed.

4.3 Predicate Poles

Beyond remarks expressing the validity of the ab-notation for predicate logic, there is essentially one passage in Wittgenstein's early writings concerning the ab-notation of predicate logic, NL, p. 95f.:

> The *ab*-notation makes it clear that *not* and *or* are dependent on one another and we can therefore not use them as simultaneous indefinables. Same objections in the case of apparent variables to old indefinables, as in the case of molecular functions: The application of the *ab*-notation to apparent–variable propositions becomes clear if we consider that, for instance, the proposition "for all x, φ x" is to be true when φ x is true for all x's and false when φ x is false for some x's. We see that *some* and *all* occur simultaneously in the proper apparent variable notation.
>
> The notation is:
> for (x) φx : a – (x) – a φx b – ($\exists$ x) – b and
> for ($\exists$x) φ x : a – ($\exists$x) – a φx b – (x) – b
>
> Old definitions now become tautologous.

Negation and disjunction are introduced as the "primitive ideas" of propositional logic in PM, p. 6-12. The first sentence of the quotation argues against these "old indefinables." Wittgenstein's objection is not based on the intention to replace $\neg$ and $\vee$ as indefinable by some other sentential connective, such as $|$. Rather, he argues against the idea of conceiving of any sentential connective as primitive. According to Wittgenstein, negation and disjunction are ab-operations, and it is nonsensical to specify primitive ab-operations because they are all internally related. The fourteen ab-operations can be constructed systematically by successively assigning the columns of poles: $aaab$, $aaba$, $abaa$, $baaa$, $aabb$, $abab$, $baab$, $abba$, $baba$, $bbaa$, $abbb$, $babb$, $bbab$, $bbba$ to the four pairs aa, ab, ba, bb. Thus, the sentential connectives and ab-operations build up a system definable by a rule. This "expression of a rule" that defines the system of ab-operations takes the place of "primitive ideas" (TLP 5.476). In the system of ab-operations, all elements depend on each other. Yet, Wittgenstein claims that primitive ideas must be independent from each other (TLP 5.451). Repeatedly, he objects to the "old indefinables", that is, the primitive ideas of PM, by referring to their cross-definability, cf. NL, p. 101[3]:

> That "or" and "not" etc. are not relations in the same sense as "right" and "left" etc., is obvious to the plain man. The possibility of cross-definitions in the old logical indefinables shows, of itself, that these are not the right indefiniables, and, even more conclusively, that they do not denote relations.

Again, the symbolic representation of truth operations in terms of function symbols obscures the fact that they compose a system. Only by representing the sentential connectives as ab-operations can the internal relations be expressed syntactically.

Due to the symbolic representation of the sentential connectives as ab- operations, not only the conditions of truth, but also the conditions of falsehood of the formulae are represented by the ab-notation. Wittgenstein refers to this in the first sentence of the above quotation by stating that "the ab-notation makes it clear that *not* and *or* are dependent on one another". This becomes clear by the paraphrase of the b-pole-group $b - \{b - p, b - q\}$ as part of the ab-symbol $p \vee q$, cf. p. 134: "$p \vee q$ is false iff p is false (= not true) and q is false (= not true)" or concisely, "$p \vee q$ is false iff not p and not q". The conditions of falsehood of disjunction are expressed by negation and conjunction. Due to the representation of the b-pole-groups, the ab-notation clarifies that disjunction and negation are not independent of each other. Negation is involved in the representation of disjunction, in so far as the explication of the conditions of falsehood of disjunction must refer to it. Presuming that primitive ideas have to be independent, negation and disjunction cannot be primitive ideas because they depend on each other as the ab-notation demonstrates.

In addressing the primitive ideas of PM in predicate logic, Wittgenstein puts forward the same objection that he levelled against the primitive ideas of PM in propositional logic. Section B, Part I of PM deals with the "Theory of apparent variables" (cf. also NL, p. 106[2]). The introductory section *9 of PM introduces $\forall x\phi x$ and $\exists x\phi x$ as the two new primitive ideas of the theory of apparent variables. These "old indefinables" are the object of Wittgenstein's criticism in the above quoted passage NL, p. 95f. Like his criticism of the "old indefinables" in propositional logic, his objection is based upon the ab-notation. The ab-notation reveals that universal and existential quantification are not independent of each other. Again, this becomes clear from the pole notation that represents not only the conditions of truth, but also the conditions of falsehood. This implies that one must refer to "some" by explicating the conditions of falsehood, if one refers to "all" by explication the conditions of truth and v.v. This becomes clear in the ab-notation. Instead of the predicate formula $\forall x\phi x$, which does not contain the existential quantifier, the corresponding expression in the ab-notation is $a - \forall x - a\phi xb - \exists x - b$. This not only contains the universal quantifier, it also contains the existential quantifier. The same holds for $\exists x\phi x$; while the formula does not contain the universal quantifier, the corresponding ab-diagram $a - \exists x - a\phi xb - \forall x - b$ does contain the universal quantifier. The simultaneous occurrence of $\exists$ and $\forall$ in a "proper apparent variable notation," explicating the conditions of truth and falsehood of predicate formulae, demonstrates that the quantifiers are, in fact, not primitive ideas.

However, Wittgenstein does not suggests analysing predicate formulae as truth functions of atomic propositions based on the fact that universal and existential quantification are not primitive ideas of the ab-notation. On the contrary, quantifiers are not removed from the ab-notation, and the truth conditions of predicate formulae are explicated by use of "all" and "some" in the paraphrase of the ab-notation, as Wittgenstein's example shows. Contrary to his explanation of the general form of truth function in TLP, he defines predicate formulae as truth functions of expressions containing quantifiers by the ab-notation. According to his explication of $\forall x\phi x$ by the ab-notation in NL, this predicate formula expresses not a truth function of atomic propositions, but of itself (like atomic propositions are truth functions of themselves, cf. TLP 5). Thus, there are un-analysable predicate formulae in predicate logic, and the task is to distinguish them from those that are analysable and make such a distinction plausible.

While sentential connectives of predicate formulae are reproduced by ab--operations, quantifiers of the formulae are retained in ab-notation. However, each quantifier of a predicate formula is replaced by two quantifiers, which are provided with a pole. By constructing the ab-diagram from the inside to the outside, the quantifier from the predicate formula provided with an a-pole is assigned to the a-pole of the constructed ab-diagram, while the other quantifier provided with the b-pole is assigned to the b-pole. The ab-diagram of $\forall x\phi x$, for example, is constructed by first providing ϕx with inmost poles, $a\phi xb$, and then assigning $a - \forall x$ with a hyphen to the a-pole and $b - \exists x$ to the b-pole: $a - \forall x - a\phi xb - \exists x - b$. Analogously, one can derive $a - \exists x - a\phi xb - \forall x - b$ for $\exists x\phi x$. Converting these ab-diagrams in pole-groups yields the following for $\forall x\phi x$:

$$
\begin{aligned}
&a - \{\forall x - a - \phi x\},\\
&b - \{\exists x - b - \phi x\}.
\end{aligned}
$$

Wittgenstein's explanation of the conditions of truth and falsehood of $\forall x\phi x$ is a straightforward paraphrase of these pole-groups: "'for all x, ϕx' is true when ϕx is true for all x's and false when ϕx is false for some x's". For $\exists x\phi x$, one receives the following pole-groups:

$$
\begin{aligned}
&a - \{\exists x - a - \phi x\},\\
&b - \{\forall x - b - \phi x\}.
\end{aligned}
$$

$\exists x\phi x$ is true iff ϕx is true for some x and false iff ϕx is false for all x's.

In contrast to the complex poles of propositional logic, the complex poles of predicate logic do not only contain propositional functions provided with a pole, they also contain quantifiers. This is a crucial difference from the method of truth tables, which does not contain a counterpart to such complex poles.

At the end of the above quotation, Wittgenstein remarks "Old definitions now become tautologous". This refers to definitions *9.01 – *9.08 of PM, p. 135. Here, negation and disjunction are defined in the realm of predicate logic. Definitions *9.01 and *9.02 concern negation:

$$
\begin{aligned}
&*9.01 \quad \sim (x).\phi x. = .(\exists x). \sim \phi x \quad Df\\
&*9.02 \quad \sim (\exists x).\phi x. = .(x) \sim \phi x \quad Df
\end{aligned}
$$

The negation sign preceding a quantifier on the left hand side of the equation is the definiendum. In the definiens on the right hand side of the equation, the negation sign occurs in the scope of the quantifiers directly preceding the propositional function. In this context, this is already introduced in section A of PM. According to chapter II of PM, it cannot be presumed that negation preceding quantifiers has the same meaning as negation preceding propositional functions, because $(x).\phi x$ is not of the same type as ϕx, cf. PM, p. 133. Thus, the meaning of negation preceding quantifiers must be defined at first. By definitions *9.01 and *9.02, negation preceding quantifiers is reduced to negation preceding propositional functions. The fact that negation should count as a primitive idea on the one hand, yet has to be re-introduced in context of predicate logic, contradicts Wittgenstein's criteria of primitive ideas. In TLP 5.451, he says with reference to negation:

> If a primitive idea has been introduced, it must have been introduced in all the combinations in which it ever occurs. It cannot, therefore, be introduced first for *one* combination and later re-introduced for another. For example, once negation has been introduced, we must understand it both in propositions of the form '$\sim p$' and in propositions like '$\sim (p \vee q)$', '$(\exists x). \sim fx$', etc. We must not introduce it first for the one class of cases and then for the other, since it would then be left in doubt whether its meaning were the same in both cases, and no reason would have been given for combining the signs in the same way in both cases.

This objection against Old Logic cannot be put forward against New Logic. Negation is depicted in the ab-diagrams by the same ab-operation in all cases and definiendum and definiens of PM *9.01 and *9.02 are represented by the same ab-symbol. Thus, their identity in meaning follows from their symbolic representation. The ab-diagrams of $\neg\forall x\phi x$ and $\exists x\neg\phi x$ are the following:

$$b - a - \forall x - a\phi xb - \exists x - b - a \qquad b - \forall x - b - a\phi xb - a - \exists x - a$$

Merely applying the rule of transitivity already leads to identical pole-groups:

$$a - \{\exists x - b - \phi x\},$$
$$b - \{\forall x - a - \phi x\}.$$

Using the ab-notation, identical ab-symbols are assigned to equivalent formulae. Using the symbolic representation, it becomes clear that there is no relevant meaningful difference between the formulae. The same holds for Def. *9.02.

By definition *9.03 – *9.08 of PM, disjunction is introduced in predicate logic. After a conversion to a modern notation, the definitions are as follows:

$$\begin{array}{lrcll}
*9.03 & \forall x\phi x \vee p & = & \forall x(\phi x \vee p) & Df \\
*9.04 & p \vee \forall x\phi x & = & \forall x(p \vee \phi x) & Df \\
*9.05 & \exists x\phi x \vee p & = & \exists x(\phi x \vee p) & Df \\
*9.06 & p \vee \exists x\phi x & = & \exists x(p \vee \phi x) & Df \\
*9.07 & \forall x\phi x \vee \exists y\psi y & = & \forall x\exists y(\phi x \vee \psi y) & Df \\
*9.08 & \exists y\psi y \vee \forall x\phi x & = & \forall x\exists y(\psi y \vee \phi x) & Df
\end{array}$$

Based on these definitions, disjunction between predicate formulae is reduced to disjunction in the scope of quantifiers. Russell and Whitehead emphasize this after the definitions, PM, p. 136:

> In virtue of these definitions, the true scope of an apparent variable is always the whole of the asserted proposition in which it occurs, even when, typographically, its scope appears to be only part of the asserted proposition. Thus when $(\exists x).\phi x$ or $(x).\phi x$ *appears* as *part* of an asserted proposition, it does not really occur, since the scope of the apparent variable really extends to the whole asserted proposition.

This point of view shall be called the standpoint of the *priority of prenex normal forms*.

In the framework of ab-notation, one confronts the problem of representing definiendum and definiens in each of the definitions *9.03 – *9.08 by the same ab-symbol. There is no hint in the received writings of Wittgenstein that describes how to solve this problem in principle. In the following, we refer to one of the above definitions, namely *9.08, to discuss the problems that arise in the framework of the ab-notation and to sketch a principal strategy for their solution. Figure 4.7 is the ab-diagram of the definiendum (= left hand side of the equation) of *9.08; figure 4.8 is the ab-diagram of the definiens (= right hand side of the equation) of *9.08.

The rule of transitivity does not suffice in this case to yield the ab-symbol from the ab-diagram. In particular, the make-up of the complex poles of the pole-groups is unclear. However, we suggest taking up the converse point of view

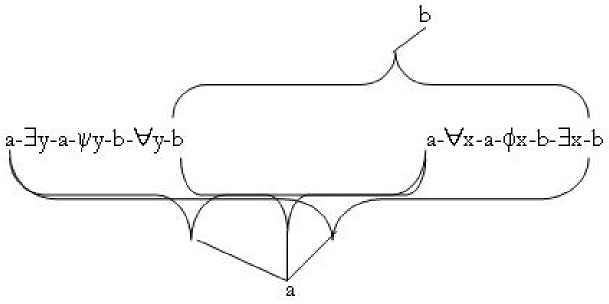

Figure 4.7: ab-diagram of $\exists y \psi y \vee \forall x \phi x$

of PM and calling for a *priority of disjunctive normal forms*, instead of a priority of prenex normal forms.[12] This generates analogous ab-symbols in predicate logic and in propositional logic. The enumeration of a-pole-groups, as well as the enumeration of b-pole-groups, can be converted to disjunctive normal forms that are equivalent to the initial formula A in case of the a-pole-groups and its negation $\neg A$ in case of the b-pole-groups. Each a-pole-group identifies a sufficient truth condition, and all a-pole-groups together identify a necessary truth condition, of A. The same holds for the b-pole-groups regarding $\neg A$. Thus, each a-pole-group represents a disjunct of a disjunctive normal form of A, and each b-pole-group represents a disjunct of a disjunctive normal form of $\neg A$. Contrary to the viewpoint of PM and Old Logic, one does not concede priority to maximize the scope of quantifiers. Conversely, New Logic calls for a minimization of the quantifiers' scope. Thus, by translating the procedure of converting ab-diagrams to pole-groups into an equivalence transformation to obtain certain predicate disjunctive normal forms, we apply commonly used PN-laws in order to generate prenex normal forms in the converse direction. Converting ab-diagrams to pole-groups by making use of a procedure that minimizes the scope of quantifiers is the fundamental strategy taken in this book to define a procedure that defines pole-groups in predicate logic, cf. section 6.2. The rules of such a procedure are part of the rules that identify the symbolizing properties of ab-diagrams. For the two ab-diagrams of figure 4.7 and figure 4.8, it suffices to make use of the

---

[12] Landini (2007), p. 117 overlooks that Wittgenstein criticizes PM's theory of quantification in NL, p. 96. There is no reason to assume that Wittgenstein was "captivated by the results of *9". Instead, if one takes into account (i) that the aim of the ab-notation is to represent logical equivalents by the same representation and (ii) that the ab-notation does not imply any reduction to propositional logic, one must deviate from the theory of quantification in PM.

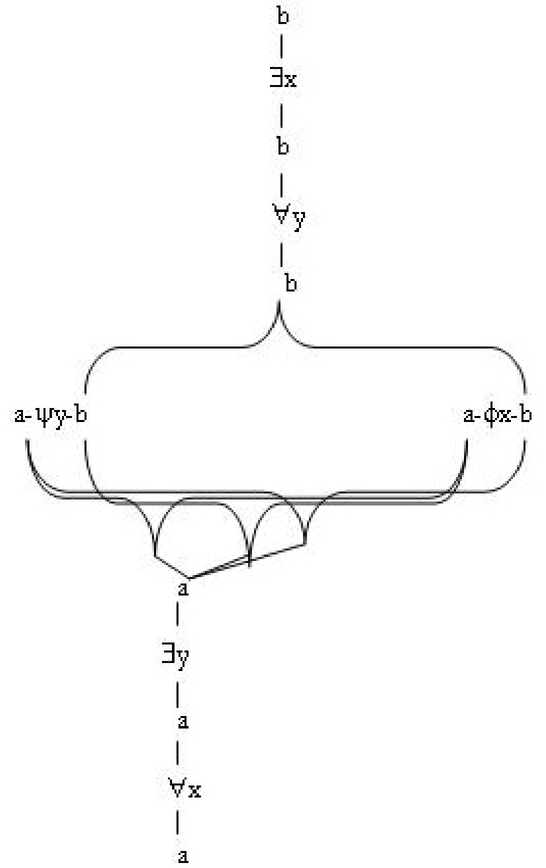

Figure 4.8: ab-diagram of $\forall x \exists y (\psi y \vee \phi x)$

rule that only the relations of a quantifier to propositional functions containing the variables bound by the quantifier are significant. As a result, we derive the same complex poles and the same pole-groups by converting the two ab-diagrams of figure 4.7 and figure 4.8:

$$
\begin{aligned}
&a - \{\exists y - a - \psi y, \quad \forall x - a - \phi x\},\\
&a - \{\exists y - a - \psi y, \quad \exists x - b - \phi x\},\\
&a - \{\forall y - b - \psi y, \quad \forall x - a - \phi x\},\\
&b - \{\forall y - b - \psi y, \quad \exists x - b - \phi x\}.
\end{aligned}
$$

Because of this transformation, we derive the same kind of complex poles that are derived by converting the formulae $\forall x \phi x$ and $\exists x \phi x$. Conceding priority to prenex normal forms would make it impossible to symbolize the dependence of complex formulae on un-analysable formulae. Mainly, it would be impossible to derive ab-symbols in which all non-symbolizing properties of ab-diagrams are eliminated. This can be seen in the formulae on the left and right hand sides of definition *9.08, and in the formula $\exists y \forall x(\psi y \vee \phi x)$, which are all equivalent. In the cases of $\exists y \forall x(\psi y \vee \phi x)$ and $\forall x \exists y(\psi y \vee \phi x)$, the ordering of quantifiers is insignificant, though this does not hold generally. Forming prenex normal forms, it is impossible to distinguish significant and insignificant orderings of quantifiers in the truth conditions of the formulae. On the contrary, by minimizing the scope of quantifiers, the syntax expresses that the ordering of the quantifiers $\exists y$ and $\forall x$ in $\exists y \forall x(\psi y \vee \phi x)$ and $\forall x \exists y(\psi y \vee \phi x)$ is not a symbolizing property. Neither of the two quantifiers is in the scope of the other in the pole-groups or the corresponding disjunctive normal forms. That the ordering of $\exists y$ and $\forall x$ in the initial formulae is insignificant is identified syntactically by reducing them to expressions in which neither $\forall x$ is in the scope of $\exists y$ nor $\exists y$ is in the scope of $\forall x$. To solve the equivalence problem by eliminating insignificant syntactical variations, the standpoint of the priority of prenex normal forms must be replaced by the standpoint of the priority of disjunctive normal forms. The latter characterizes New Logic, the former Old Logic.

However, in the following systematic elaboration of the ab-notation, we initially leave out the problem of minimizing the quantifier's scope by considering only formulae of "elementary predicate logic" in chapter 5. Elementary predicate logic we call the logic of the subclass of predicate formulae that does not contain dyadic sentential connectives in the scope of quantifiers. Due to this limitation, the problem of converting ab-diagrams to pole-groups can be solved for this subclass of predicate formulae merely by applying Wittgenstein's rule of transitivity. However, another problem is addressed first: the problem of converting pole-groups to ab-symbols. As indicated on page 129, merely converting formulae in pole-groups is not sufficient to solve the equivalence problem, even in propositional logic. The problem of converting different pole-groups of equivalent formulae to identical pole-groups, i.e. their ab-symbols, shall be solved by a procedure involving minimizing single pole-groups. Similar procedures are known in propositional logic, e.g. the Quine- McCluskey algorithm, cf. section 5.4.1.1. Minimizing the number of pole-groups is not the goal because, even in propositional logic, the problem of minimizing disjunctions has no unique solution given canonical disjunctive normal forms CDNF. However, the problem of minimizing *single* disjuncts has a unique solution. Referring to the Quine-McCluskey algorithm, the equivalence problem can be solved in propositional logic by generating the so called “reduced disjunctive normal forms” RDNF. While CDNF only contain minterms, RDNF solely contain prime implicants, cf. section 5.4.1.1. Roughly speaking, the a-pole-groups generated from ab-diagrams, correspond to the CDNF of an initial formula A, while the b-pole-groups correspond to the CDNF of $\neg A$. The procedure of minimizing pole-groups imitates the construction of RDNF according to the Quine-McCluskey procedure. This brings about problems for handling predicate logic within New Logic. These shall be explained in the following by slightly modifying the formula of the left hand side of *9.08 such that in both disjuncts the same propositional function occurs:

$$\exists y \phi y \vee \forall x \phi x. \tag{4.1}$$

This formula is equivalent to:

$$\exists y \phi y. \tag{4.2}$$

The pole-groups of formula (4.2) are, cf. above p. 142:

$$\begin{aligned} &a - \{\exists y - a - \phi y\}, \\ &b - \{\forall y - b - \phi y\}. \end{aligned}$$

$\exists y \phi y$ is also equivalent to $\exists x \phi x$. According to Wittgenstein’s terminology, this means that the specific type of the variable is not a symbolizing property. Moreover, the relationship between the bound variable and its occurrences in argument positions of propositional functions is the symbolizing property. To eliminate non-symbolizing properties, the specific variables shall be replaced by numbers. This will be explained in detail on p. 160. The pole-groups of $\exists y \phi y$, i.e. (4.2), as well as of $\exists x \phi x$ are:

$$\begin{aligned} &a - \{\exists_1 - a - \phi_1\}, \\ &b - \{\forall_1 - b - \phi_1\}. \end{aligned}$$

However, by replacing variables with numbers and applying the rule of transitivity, one obtains the following pole-groups of (4.1):

$$a - \{\exists_1 - a - \phi_1, \quad \forall_1 - a - \phi_1\},$$
$$a - \{\exists_1 - a - \phi_1, \quad \exists_1 - b - \phi_1\},$$
$$a - \{\forall_1 - b - \phi_1, \quad \forall_1 - a - \phi_1\},$$
$$b - \{\forall_1 - b - \phi_1, \quad \exists_1 - b - \phi_1\}.$$

Thus, one still does not yield identical pole-groups for the equivalent formulae (4.1) and (4.2), even by replacing variables with numbers and applying the rule of transitivity. To define rules that minimize the pole-groups of $\exists y \phi y \vee \forall x \phi x$, i.e. (4.1), such that one derives the same pole-groups as with the equivalent formula $\exists y \phi y$, i.e. (4.2), internal relations between complex poles must be considered. Thus, for example, $\exists_1 - a - \phi_1$ and $\forall_1 - b - \phi_1$ are contradictory. Therefore they are contrary and subcontrary, too. The same holds for $\forall_1 - a - \phi_1$ and $\exists_1 - b - \phi_1$. Furthermore, $\forall_1 - b - \phi_1$ and $\forall_1 - a - \phi_1$ are contrary. Thus, the third a-pole-group can be eliminated, and the merging rule used to generate the prime implicants in the Quine-McCluskey algorithm (cf. p. 207) can be applied to the first two a-pole-groups. Thus, we obtain $a - \{\exists_1 - a - \phi_1\}$ as the remaining a-pole-group. Finally, by considering that the complex pole $\forall_1 - b - \phi_1$ implies the complex pole $\exists_1 - b - \phi_1$, one can reduce the pole-groups of (4.1) to the pole-groups of (4.2) by applying plain equivalence rules (cf. p. 215). For now, it suffices to point out that two main problems arise from the conversion of pole-groups to ab-symbols: (i) minimizing pole-groups by referring to internal relations between complex predicate poles – the known rules of propositional logic being merely a special case of these minimization rules; (ii) defining the internal relations of complex predicate poles in general by solely referring to their structural, syntactical properties. The solution of these two problems in the realm of elementary predicate logic will be elaborated in the following chapter. This will demonstrate the solution of the equivalence problem by means of the ab-notation for a subclass of predicate logic expanding propositional logic. In chapter 6 we will then generalize this solution in the whole realm of predicate logic.

The internal relations of complex poles in propositional logic are trivial; poles of form $a - \xi$ and $b - \xi$ are contradictory. In addition, these complex poles are logically independent of each other. Furthermore, the complex poles can be identified directly from the propositional formula. They are already determined by the occurrences of the propositional variables in the propositional formula. Consequently, all their possible 2^n combinations can be constructed immediately. This is essentially what happens in the left part of a truth table, as well. Truth tables contain 2^n combinations of n different propositional variables occurring in the propositional formula. Meanwhile, pole-groups constructed from ab-diagrams contain 2^n combinations of n *occurrences* of propositional variables. In predicate logic, however, the complex poles are not identifiable directly. They must be generated by converting ab-diagrams to pole-groups. Furthermore, their internal relations are not limited to contradictory relations, but are far more complex, including subcontrary, contrary, and subaltern relations. Taking this into account, it becomes intelligible why, contrary to truth tables, ab-notation is based on the occurrences of propositional variables and why combinations of contradictory propositional complex poles are allowed for, cf. p. 123. Allowing for combinations of contradictory poles, and more generally dispensing with consideration of internal relations by constructing ab-diagrams, is motivated by the intention to establish a notation for the whole realm of predicate logic. The rules of the ab-notation in propositional logic rely on an important insight. This is that the program of New Logic is only realizable in predicate logic if the general problem of identifying internal relations between formula can be reduced to the problem of identifying internal relations between complex poles and groups containing them. *Reducing the identification of internal relations of predicate formulae to internal relations between pole-groups* is a further attribute of New Logic. On the other hand, Old Logic does not have a comparable concept of complex predicate poles (predicate literals). As will be seen in the following, Wittgenstein's programme of a New Logic depends first and foremost on the possibility of identifying internal relationships between pole-groups based solely on their structural syntactical properties.

4.4 Summary of Part I

In Part I, we contrasted the general concept of New Logic and traditional mathematical logic, and we discussed the principal ideas behind Wittgenstein's programme by taking his remarks on his ab-notation as a starting point.

Whereas Old Logic is characterized by the concept of an axiomatic proof, New Logic aims for *iconic proofs*. A logical proof, according to New Logic, consists of a mechanical reduction of syntactical differences between equivalent formulae. Such a reduction converts all equivalent formulae to a single symbol that identifies the truth conditions of the initial formulae based on its syntactical properties. This proof conceptualization is realized by solving the *equivalence problem*. We argued that New Logic must be understood against the background of Wittgenstein's *intensional conception* of logic and arithmetic. This means (i) that he objects to arbitrary functions or sets as fundamental concepts in favour of his notion of *operations*, and (ii) that he claims that logical and arithmetical problems should be solved by inventing *ideal notations* that allow for a solution of the problems based on their syntactical features alone. The aim of this conceptualization is to create a strictly *pure syntactical foundation* of logic and arithmetic that does not even allow for derivations from axioms that cannot be proven by syntactical means. Instead, any logical or arithmetical property must be identified by the syntactical features of its adequate formal representation. From this, several negative and positive consequences can be drawn. On the negative side, it follows that Wittgenstein rejects *p.r. functions* as adequate formal representations of logical or arithmetical properties. Furthermore, he rejects basing adequate formalization on the notions of *expressing* and *capturing* a formal property. Thus, he also objects to *Church's thesis* and to the method of *diagonalization*, as found in the proofs of *Church's theorem* and *Cantor's theorem*. According to Wittgenstein's point of view, these so-called proofs are fallacies, as are the structural analogous paradoxes of *Richard's Paradox* or the *Liar's Paradox*, because they all rest on an inability to distinguish between operations and functions. These fallacies are not avoided by distinguishing *meta- and object-language*, as this distinction itself does not distinguish between operations and functions. Furthermore, his intensional conceptualization abandons any *logical formalization of arithmetic* and an *extensional understanding of infinity*. Instead, he invents his Ω-notation for arithmetic and claims that any infinitude must be defined by means of operations. We have illustrated this point through his rejection of "*arithmetical experiments*", which define an infinite series of numbers by deciding on their progression individually. According to Wittgenstein's standards, this process never "leads to the totality," but only allows the determination of finite sequences. Furthermore, we noted that his demand to define infinite series of numbers by operation essentially rests on the representation of numbers by a notation "that allows the recognition of a law," such as the notation of continued fractions in the case of $\sqrt{2}, e$, or π, rather than their decimal notation.

We applied Wittgenstein's rejection of any extensional attitude to infinity to the programme of New Logic. We specified the problem of semantics as the problem of defining the totality of models and counter-models in the realm of logic, without deciding for each interpretation whether it is a model or a counter-model. We also put forward that the problem of constructing the totality of all relations of implications by iterative application of operations alone (problem of implication) arises similarly in the realm of predicate logic. We not only showed certain problems, such as the *problem of semantics* and the *problem of implication* as well as the above mentioned *problem of equivalence* within New Logic, but also presented strategies to solve them. Wittgenstein's conception is essentially connected to the invention of a proper notation. We argued that the *ab-notation* serves as proper notation within predicate logic. By referring to Wittgenstein's remarks on the ab-notation of propositional logic, we illustrated his criticism of Old Logic that confuses operations and functions through his representation of logical particles, such as $\neg, \wedge, \vee, \rightarrow$. He represents these by means of *ab-operations*, instead of using them as symbols of functions. Furthermore, we explained his fundamental notion of a *symbolizing property* by referring to syntactical features of ab-notation that are significant for identifying the formulae's truth conditions. Wittgenstein's *rule of transitivity* that results in his *pole-group notation* illustrates how to generate *ab-symbols* which are deprived of insignificant syntactical features. The general problem of the ab-notation requires defining sufficient rules for constructing ab-symbols, in addition to Wittgenstein's rule of transitivity in propositional logic, and to expand this solution to predicate logic. By discussing Wittgenstein's extension of ab-notation to predicate logic, we argued that this notation is resistant to what Wittgenstein later called "the *biggest mistake of TLP*," namely reducing predicate logic to propositional logic. On the contrary, quantifiers are an essential, irreducible part of ab-notation, and they are used to determine the truth conditions of predicate formulae. Furthermore, by contrasting the conception of New Logic to *Principia Mathematica*, we suggested that *priority* must be given to *distributive normal forms* in predicate logic, rather than prenex normal forms. This is necessary to generate ab-symbols of predicate logic that explain the truth conditions of predicate formulae. As a result, an *equivalent to atomic propositions* of propositional logic must be defined for predicate logic – *predicate poles* in the ab-notation or their counterpart in predicate logic, namely *closed structures*. Finally, we indicated that the equivalence problem, as well as the problem of defining the totality of internal relations of predicate logic by operations (including the problem of implication), is *reducible* to the problem of defining a *system of implications between these distributive normal forms* in predicate logic.

Part III

Systematic elaboration

Chapter 5

Elementary predicate logic

This chapter describes the solution of the equivalence problem using the ab-notation for elementary predicate logic. Elementary predicate logic is a subclass of predicate logic that does not contain dyadic sentential connectives in the scope of quantifiers. We distinguish elementary predicate logic from "molecular predicate logic," which refers to the entire class of predicate logic that also contains dyadic sentential connectives in the scope of quantifiers. At first, we limit the ab-notation to elementary predicate logic to apply the program of New Logic to the realm of predicate logic without making matters too complicated. Before discussing how the ab-notation is applied to the whole realm of pure predicate logic, a task undertaken in the next chapter, in this chapter, we will show how to expand the ab-notation from propositional logic to a subclass of predicate logic. Thus, the rules of the ab-notation are developed by stepwise generalization.

Whereas the PM notation was primarily used in chapter 4, we now rely on the common notation of pure predicate logic. Thus, we will use capital letters P, Q, R, S, T, U and $P_1, P_2, \ldots$ as propositional variables; F, G, H, I, J and $F_1, F_2, \ldots$ as predicate letters; $x, x_1, x_2, \ldots; y, y_1, y_2, \ldots; z, z_1, z_2, \ldots$ as variables; a, b, c, d and $a', a'', \ldots$ as names; $\neg$, $\wedge$, $\vee$, and $\rightarrow$ as sentential connectives; $\forall$ and $\exists$ as quantifiers and parentheses $(,)$. As we only refer to pure predicate logic, we will neither make use of identity nor mathematical functions. Superscript strokes will be used as indices for names to avoid ambiguities in the notation of the symbolizing poles in ab-notation (cf. section 5.2). We will make use of the following rules for parentheses: (i) outermost parentheses are eliminated, (ii) $\wedge$ binds stronger than $\vee$, $\wedge$ and $\vee$ bind stronger than $\rightarrow$, (iii) instead of $((A \wedge B) \wedge C), ((A \vee B) \vee C), ((A \rightarrow B) \rightarrow C)$, we write $A \wedge B \wedge C, A \vee B \vee C, A \rightarrow B \rightarrow C$, respectively ($A, B, C$ are meta-variables for parts of predicate formulae).

Any formula of predicate logic satisfying these conditions is abbreviated by "*wff*" (= well-formed formula).

5.1 Pole-groups

The equivalence problem is solved by assigning the same ab-symbol to every *wff* of a class of equivalent formulae, and assigning different ab-symbols to non-equivalent *wff*s. This assignment is carried out by the following steps:

$$wff \Rightarrow \text{ab-diagram} \Rightarrow \text{pole-groups} \Rightarrow \text{ab-symbol}$$

These steps are carried out by constructing the expression of each step from the expression of the previous step. By constructing the ab-diagram from a *wff*, we establish that the innermost a- and b-poles are connected to propositional functions, and the connections are made by "edges" ($-$). Meanwhile, all other a- and b-poles are connected to poles or quantifiers of the constructed diagram, and the connections are made by "hyphens" (-). Quantifiers are connected to a- and b-poles in the process of constructing ab-diagrams by edges. Edges and hyphens are distinguished in order to identify constituents of complex poles. In addition to propositional functions, only those parts that are added by edges in the construction of ab-diagrams are constituents of complex poles. Furthermore, it is presumed that propositional functions are first provided with the innermost poles connected by edges, and then, in addition, with the outer poles connected by hyphens. For example, the ab-diagram of P is $a\text{-}a - P - b\text{-}b$, and not simply $a - P - b$. Because of this, every ab-diagram is provided with innermost and outermost a- and b-poles. Only the innermost poles are parts of complex poles, while the outermost poles are assigned to pole-groups. These conventions make it easier to identify complex poles as the elements of pole-groups in the ab-diagrams.

Given this, any *wff*, i.e. formulae of elementary as well as of molecular predicate logic, is converted to an ab-diagram by the following rules:

translation rule: To convert a *wff* A to an ab-diagram, one must carry out the following steps from the inside to the outside according to the logical hierarchy of A:

propositional function rule: Propositional functions must be provided with an innermost a-pole to the left and an innermost b-pole to the right.

sentential connective rule: Sentential connectives must be translated according to their definitions as ab-operations.

quantifier rule: A quantifier of A, together with the bound variable μ, must be connected to the outermost a-pole of the constructed ab-diagram. The opposed quantifier in each case, together with μ, must be connected to the b-pole.

pole rule: An a-pole must be connected to an innermost a-pole, and a b-pole must be connected to an innermost b-pole. Furthermore, an a-pole must be connected to a quantifier if the quantifier is connected to an a-pole. A b-pole must be connected to a quantifier if the quantifier is connected to a b-pole.

edge rule: Innermost poles are connected to propositional functions by edges, and quantifiers are also connected to poles by edges.

hyphen rule: Outer poles are connected to poles by hyphens, and pairs of poles are combined by curly brackets.

termination rule: The ab-diagram is constructed completely if all signs of A are translated and the ab-diagram contains an outermost a- and an outermost b-pole.

Converting ab-diagrams to pole-groups is trivial in elementary predicate logic. A "path" is the concatenation of poles and quantifiers leading, via the curly brackets and intermediate poles, from an outermost pole of the ab-diagram to the innermost poles of propositional functions. These paths branch out if a pole is connected with pairs of poles by *several* brackets. Each path runs over one branch. Quantifiers occur only inside brackets in the ab-diagrams of elementary predicate logic. That is why paths do not branch out any more after having reached quantifiers. Each pole-group is gained from one path leading from an outermost pole to innermost poles. The a-pole-groups are taken from paths starting with the outermost a-pole; the b-pole-groups are taken from those paths starting with the outermost b-pole. All intermediary poles and their hyphens on these paths, as well as all intermediary edges, are to be eliminated. Thus, the outermost pole plus quantifiers, which are connected by an edge to innermost poles of propositional functions, remain for each path. A path without its intermediary poles, hyphens, and edges is called an "elementary path". A pole-group corresponds to each elementary path. This procedure is based on Wittgenstein's rule of transitivity. In molecular predicate logic, this rule no longer suffices. It will be replaced by a more general rule in cf. section 6.2. However, the following concise rule suffices when referring to the identification of elementary paths in elementary predicate logic:

path-rule: In order to convert an ab-diagram to pole-groups, all elementary paths must be generated. Each elementary path starting with the a-pole makes up an a-pole-group; each elementary path starting with the b-pole makes up a b-pole-group.

EXAMPLE 1: $\forall x \neg \exists y \neg Fxy$

Step 1 (*propositional function rule* and *edge rule*):

$$a - Fxy - b$$

Step 2 (*pole rule* and *hyphen rule*):

$$a\text{-}a - Fxy - b\text{-}b$$

Step 3 (*sentential connective rule* and *hyphen rule*):

$$b\text{-}a\text{-}a - Fxy - b\text{-}b\text{-}a$$

Step 4 (*quantifier rule* and *edge rule*):

$$\forall y - b\text{-}a\text{-}a - Fxy - b\text{-}b\text{-}a - \exists y$$

Step 5 (*pole rule* and *hyphen rule*):

$$b\text{-}\forall y - b\text{-}a\text{-}a - Fxy - b\text{-}b\text{-}a - \exists y\text{-}a$$

Step 6 (*sentential connective rule* and *hyphen rule*):

$$a\text{-}b\text{-}\forall y - b\text{-}a\text{-}a - Fxy - b\text{-}b\text{-}a - \exists y\text{-}a\text{-}b$$

Step 7 (*quantifier rule* and *edge rule*):

$$\forall x - a\text{-}b\text{-}\forall y - b\text{-}a\text{-}a - Fxy - b\text{-}b\text{-}a - \exists y\text{-}a\text{-}b - \exists x$$

Step 8 (*pole rule* and *hyphen rule*) results in the ab-diagram:

$$a\text{-}\forall x - a\text{-}b\text{-}\forall y - b\text{-}a\text{-}a - Fxy - b\text{-}b\text{-}a - \exists y\text{-}a\text{-}b - \exists x\text{-}b$$

Application of the *path rule* results in the pole groups:

$$a - \{\forall x \forall y - a - Fxy\},$$
$$b - \{\exists x \exists y - b - Fxy\}.$$

EXAMPLE 2: $\neg\exists x \neg\neg Fx \vee \neg\neg\forall x \neg Fx$

Step 1 (*propositional function rule* and *edge rule*):

$a - Fx - b$ $\quad$ $a - Fx - b$

Step 2 (*pole rule* and *hyphen rule*):

$a\text{-}a - Fx - b\text{-}b$ $\quad$ $a\text{-}a - Fx - b\text{-}b$

Step 3 (*sentential connective rule* and *hyphen rule*):

$a\text{-}b\text{-}a\text{-}a - Fx - b\text{-}b\text{-}a\text{-}b$ $\quad$ $b\text{-}a\text{-}a - Fx - b\text{-}b\text{-}a$

Step 4 (*quantifier rule* and *edge rule*):

$\exists x - a\text{-}b\text{-}a\text{-}a - Fx - b\text{-}b\text{-}a\text{-}b - \forall x$ $\quad$ $\exists x - b\text{-}a\text{-}a - Fx - b\text{-}b\text{-}a - \forall x$

Step 5 (*pole rule* and *hyphen rule*):

$a\text{-}\exists x - a\text{-}b\text{-}a\text{-}a - Fx - b\text{-}b\text{-}a\text{-}b - \forall x\text{-}b$ $\quad$ $b\text{-}\exists x - b\text{-}a\text{-}a - Fx - b\text{-}b\text{-}a - \forall x\text{-}a$

Step 6 (*sentential connective rule* and *hyphen rule*):

$b\text{-}a\text{-}\exists x - a\text{-}b\text{-}a\text{-}a - Fx - b\text{-}b\text{-}a\text{-}b - \forall x\text{-}b\text{-}a$ $\quad$ $b\text{-}a\text{-}b\text{-}\exists x - b\text{-}a\text{-}a - Fx - b\text{-}b\text{-}a - \forall x\text{-}a\text{-}b\text{-}a$

Step 7 (*sentential connective rule* and *hyphen rule*) results in the ab-diagram (cf. figure 5.1).

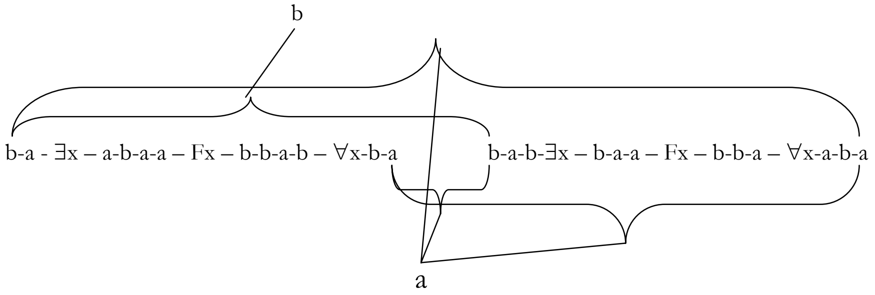

Figure 5.1: ab-diagram of $\neg\exists x\neg\neg Fx \vee \neg\neg\forall x\neg Fx$

Applying the *path rule* results in the pole-groups:

$$\begin{aligned}
&b-\{\exists x-a-Fx, \quad \exists x-a-Fx\},\\
&a-\{\exists x-a-Fx, \quad \forall x-b-Fx\},\\
&a-\{\forall x-b-Fx, \quad \exists x-a-Fx\},\\
&a-\{\forall x-b-Fx, \quad \forall x-b-Fx\}.
\end{aligned}$$

The class of all a-pole-groups of some *wff*, A, can be translated into a disjunction of conjunctions of elementary closed structures ($\bigvee\bigwedge ecs$) that is equivalent to A. Meanwhile, the class of all b-pole-groups can be translated into a $\bigvee\bigwedge ecs$ equivalent to $\neg A$. An "elementary closed structure," ecs, is a formula composed of a sequence of n quantifiers (≥ 0) and a propositional function that is preceded by at most one negation sign. For example, $\neg P$, $\forall x\exists yFxy$, $\exists x\forall y\exists z\neg Fxxyyz$ are ecs. ecs correspond to elementary poles. In elementary predicate logic, this translation is trivial. The general rules for translating pole-groups into *wff*s and converting *wff*s into equivalent $\bigvee\bigwedge cs$ are defined in sections 6.2, 6.1.3.2 and 6.1.3.3.

5.2 Symbolizing complex poles

Pole-groups constructed by applying the *path-rule* contain complex poles with non-symbolizing properties (cf. section 4.2 according to the notion "symbolizing property”). As explained on p. 148, the types of the variables are not symbolizing properties; their symbolizing properties are their relationship to the argument positions within the scope of a quantifier. According to the identification of truth conditions, it is not relevant *which* variables are used. The significance is in the use of *identical* versus *different* variables at argument positions of the propositional variables in the scope of quantifiers. This becomes clear by considering equivalent formulae that only differ in the types of variables. For example, $\forall x \exists y Fxxy$ and $\forall y \exists x Fyyx$. In this case, the labels“x” and “y” are not significant. Instead, the fact that the variable bound by the universal quantifier occurs at the first and second argument position, while the variable bound by the existential quantifier occurs at the third argument position of the propositional function, is significant. In order to abstain from the non-symbolizing specific type of variables, we replace them with numbers denoting argument places of propositional functions. Thus, each occurrence of a variable at the n-th argument place of a propositional function must be replaced by the number n. To express the relation of a bound variable to several occurrences of the variable at different argument places of a propositional functions, “closed forks” ($<$, $-$, etc.) will be used. Closed forks consist of prongs that meet in a knot.[1] Thus, the complex pole $\exists x - a - Fxx$ is represented by $\exists <^1_2 -a - F_{12}$. The closed fork denotes that the first and second argument position must be satisfied by the *same* object. The complex pole $\forall x - a - Fxx$ is represented by $\forall <^1_2 -a - F_{12}$. This means that the first and second argument positions must be satisfied by the *same* objects. The numbers connected by a fork constitute a column. The order of the numbers of a column is not a symbolizing property. The fact *that* they are connected by a closed fork is significant, whereas *how* they are connected through a closed fork is not meaningful. That is why the complex poles of elementary predicate logic, which contain only one propositional function, are to be read as single-row expressions.

Variables are to be replaced by numbers according to the following rule:

elementary variable rule: Variables are replaced by numbers. Every occurrence of a variable at the n-th argument position of a propositional function is replaced by the number n. The variable succeeding the quantifier is replaced by all the numbers that replace the variable in the propositional function. If replaced by more than one number, all the numbers are connected by a closed fork.

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[1] In addition to closed forks “open forks” ($\lessdot$, $\lessdot$ etc.) are used in molecular predicate logic, in which a quantifier can be related to several propositional functions. In these cases, the prongs do not meet in a knot. However, in elementary predicate logic, we can abstain from open forks.

Furthermore, only the order of *different* quantifiers is a symbolizing property in the complex poles of elementary predicate logic, whereas the order of identical quantifiers is not significant. $\exists_1\exists_2 - a - F_{12}$ is the same symbol as $\exists_2\exists_1 - a - F_{12}$ because, unlike $\exists x\forall yFxy$ and $\forall y\exists xFxy$, $\exists x\exists yFxy$ and $\exists y\exists xFxy$ are equivalent. For this reason, sequences of identical quantifiers in complex poles shall be symbolized as enumerations by separating single quantifiers of a sequence of identical quantifiers with commas. Hence, instead of $\exists_1\exists_2 - a - F_{12}$, we write $\exists_1,\exists_2 - a - F_{12}$. As with enumerating complex poles and pole-groups, enumerating quantifiers reveals that the order of the enumerated elements is not significant. Enumerations have no structure. Hence, no structural property is available that might serve as a symbolizing property.

quantifier ordering rule: Existential quantifiers of a sequence of existential quantifiers, and universal quantifiers of a sequence of universal quantifiers, are separated by commas.

Unlike the type of variables, the type of names is a symbolizing property. For example, Fa and Fb are not equivalent. Hence, names must be retained in the complex poles. Not only the type of names, but also their occurrence at argument positions is a symbolizing property. In order to proceed concertedly, the following rule is applied in reference to names:

name rule: Names occurring in a propositional function are enumerated. In this enumeration, single names are separated by a comma. The enumeration of names precedes the sequence of quantifiers, and it is separated from the sequence of quantifiers by a comma. If the complex pole does not contain any quantifier, the enumeration of names must be connected to the innermost pole by an edge, "–". Every occurrence of a name at the n-th argument position of a propositional function is replaced by the number n. These numbers must be connected by a closed fork that is written down subsequent to the name in the enumeration of the names.

Because of this rule and the *elementary variable rule*, all argument positions of propositional functions are denoted by numbers. Furthermore, the relation to argument positions appears in a similar form in case of names and variables.

According to the *name rule*, $\forall x - a - Fxaab$ is replaced by $a <^2_3, b_4,\forall_1 - a - F_{1234}$. Names are separated by commas because their ordering is not significant, and the same holds for the the enumeration of names in relation to the sequence of quantifiers.

We use the term "symbolizing complex poles" to denote complex poles with: (i) k numbers at the argument positions of propositional functions ($k \geq 0$), (ii) preceding sequences of names of length m ($m \geq 0$), and (iii) sequences of quantifiers with length n ($n \geq 0$). We call their respective pole-groups "symbolizing pole groups". The sequences of names and quantifiers of symbolizing poles are called the "prefix". In order to convert complex poles into symbolizing complex poles, the *elementary variable rule*, the *quantifier ordering rule*, and the *name rule* are applied.

Converting pole-groups into symbolizing pole-groups is a first, trivial step in their transformation to ab-symbols. The second, less trivial step consists of minimizing the symbolizing pole-groups (cf. section 5.4).

If we speak of "pole" or "complex poles" in the following, we refer to "symbolizing complex poles," unless we do not explicitly refer to complex poles not yet converted to symbolizing complex poles, or to the simple poles "a" and "b". Instead of referring to "symbolizing complex poles of elementary predicate logic", we will shorten the phrase to "elementary poles". For a precise syntactic definition of elementary poles, cf. p. 237.

5.2.1 Paraphrase

When we refer to a paraphrase of pole-groups, we mean their translation to ordinary language. Through this paraphrase, one comes to understand how the pole-groups identify conditions of truth and falsehood. In section 6.3.3, we will lay down the general procedure for paraphrasing symbolizing pole-groups, such that they are interpreted as descriptions of structural properties of truth conditions (models) and falsehood conditions (counter-models) of a *wff* (cf. section 6.3.1). In this section, we will merely sketch a simplified procedure of paraphrasing pole-groups in elementary predicate logic. We abstain from further explanations as the purpose of this section is only to become an understanding of the meaning and use of elementary pole-groups without considering the conception of New Semantics.

The paraphrasing of a-pole-groups of a *wff*, A, is of the form "A is true iff ..."; the paraphrasing of b-pole-groups is of the form "A is false iff ...". To clarify that a-pole-groups and b-pole-groups identify conditions of interpretations, $\Im$, in order to be models and counter-models, one can alternatively paraphrase the pole-groups by "$\Im$ is a model of A iff ..." and "$\Im$ is a counter-model of A iff ...". The paraphrases of single pole-groups are connected by "or," while the paraphrases of single complex poles are connected by "and".

The paraphrases of symbolizing complex poles remain to be explained. In propositional logic, the complex poles contain only propositional functions, with 0 argument positions and without prefixes. Propositional complex poles of form $a - \xi$ are paraphrased by "ξ is true" or, alternatively, "$\Im(\xi) = T$". Complex poles of form $b - \xi$ are paraphrased by "ξ is false" or "$\Im(\xi) = F$". Poles of monadic and polyadic elementary predicate logic identify tuples of single interpretations of predicates $\Im(\varphi)$. If the innermost pole is the a-pole, tuples must satisfy $\Im(\varphi)$ to determine if $\Im$ is a model or a counter-model. If the innermost pole is the b-pole, tuples are identified that must not satisfy $\Im(\varphi)$, to determine if $\Im$ is a model or a counter-model. Prefixes of poles identify certain *combinations of objects of the domain* that must or must not satisfy $\Im(\varphi)$. Complex poles of elementary predicate logic are paraphrased by paraphrasing the prefix from left to the right by connecting the paraphrase of the names and quantifiers with the phrase "combined with". Each name t is paraphrased with "$\Im(t)$", each existential quantifier $\exists$ with "some object", and each universal quantifier $\forall$ with "all objects". Closed forks are paraphrased with "the same". Numbers $k \dots l$ (k,$l \leq m$) succeeding names or quantifiers are paraphrased by "at the k-th and … and l-th position of the m-tuples of $\Im(\varphi)$", where m is the number of argument positions of φ. The paraphrase of the complex poles must be closed by "satisfies $\Im(\varphi)$" or, if the prefix starts with a universal quantifier, "satisfy $\Im(\varphi)$" if the innermost pole is the a-pole. Otherwise, the paraphrase is closed by "does not satisfy $\Im(\varphi)$" or "do not satisfy $\Im(\varphi)$".

EXAMPLE 1. The only a-pole-group of the *wff* $\exists x_1 \forall y \forall z F x_1 y z x_1$ is:

$$a - \{\exists <^1_4 \forall_2,\forall_3 - a - F_{1234}\}.$$

Hence, the paraphrase of the a-pole-groups of $\exists x_1 \forall y \forall z F x_1 y z x_1$ is as follows:

"$\Im$ is a model of $\exists x_1 \forall y \forall z \exists x_2 F x_1 y z x_2 x_1$ iff some object, the same at the 1. and 4. position of the 4-tuples of $\Im(F)$, combined with all objects at the 2. position of the 4-tuples of $\Im(F)$, combined with all objects at the 3. position of the 4-tuples of $\Im(F)$, satisfies $\Im(F)$."

Given the domain $\{c_1, c_2\}$, all models can be constructed with this description. The models consist of all those $\Im(F)$ that contain at least one of the following two sets of tuples:

1. $(c_1c_1c_1c_1), (c_1c_1c_2c_1), (c_1c_2c_1c_1), (c_1c_2c_2c_1),$
2. $(c_2c_1c_1c_2), (c_2c_1c_2c_2), (c_2c_2c_1c_2), (c_2c_2c_2c_2).$

EXAMPLE 2. The *wff* $\exists x \forall y \exists z F x y y z$ has the following a-pole-group:

$$a - \{\exists_1 \forall <^2_3 \exists_4 - a - F_{1234}\}.$$

Paraphrase:

"$\Im$ is a model of $\exists x \forall y \exists z Fxyyz$ iff some object at the 1. position of the 4-tuples of $\Im(F)$, combined with all objects, the same at the 2. and 3. position of the 4-tuples of $\Im(F)$, combined with some object at the 4. position of the 4-tuples of $\Im(F)$, satisfies $\Im(F)$."

Thus, those and only those $\Im$ with domain $\{c_1, c_2\}$ that contain at least one of the following eight sets of tuples in $\Im(F)$ are models:

1. $(c_1c_1c_1c_1), \quad (c_1c_2c_2c_1),$
2. $(c_1c_1c_1c_1), \quad (c_1c_2c_2c_2),$
3. $(c_1c_1c_1c_2), \quad (c_1c_2c_2c_1),$
4. $(c_1c_1c_1c_2), \quad (c_1c_2c_2c_2),$
5. $(c_2c_1c_1c_1), \quad (c_2c_2c_2c_1),$
6. $(c_2c_1c_1c_1), \quad (c_2c_2c_2c_2),$
7. $(c_2c_1c_1c_2), \quad (c_2c_2c_2c_1),$
8. $(c_2c_1c_1c_2), \quad (c_2c_2c_2c_2).$

At the 2. and 3. position, c_1c_1 and c_2c_2 occur in each set, and at the 1. position, the same object occurs in each set, whereas at the 4. position, an arbitrary object occurs in each tuple.

Section 6.3.3 specifies the general rules of paraphrasing symbolizing pole-groups. Section 6.3.4 more precisely delineates the concept of structures of interpretation $\Im$, and section 6.3.5 describes how to construct models and counter-models of any *wff* given their symbolizing pole-groups.

5.3 System of implications

The procedure of minimizing pole-groups presupposes the identification of internal relations of poles and pole-groups. This section describes how these internal relations can be identified completely by referring to their structural, syntactical properties. Here we define the "system of elementary predicate logic" and solve the problem of implication in the realm of elementary predicate logic (cf. section 3.1.7).

5.3.1 Reduction

In this section, we specify how to reduce the totality of internal relations between *wff*s of elementary predicate logic to the relations of implication between pole-groups. Internal relations in predicate logic are implication, contrariety, subcontrariety, and, as a special case of the two latter relations, contradiction. Identifying these internal relations can be reduced to identifying implications, and thus, to the problem of implication.

As any *wff* of elementary predicate logic A can be converted to its pole-groups, it can be converted to a $\bigvee\bigwedge ecs$, equivalent to A, and to a $\bigvee\bigwedge ecs$, equivalent to $\neg A$. Thus, it suffices to identify the internal relations between $\bigvee\bigwedge ecs$, or equivalently between the corresponding pole-groups, to identify the totality of internal relations of the *wff*s of elementary predicate logic.

In propositional logic, the only internal relations of poles are contradictory relations, but in predicate logic, subaltern, subcontrary, and contrary relations also exist between predicate poles. This is demonstrated by the translation of the logical square in the pole notation, cf. figure 5.2.

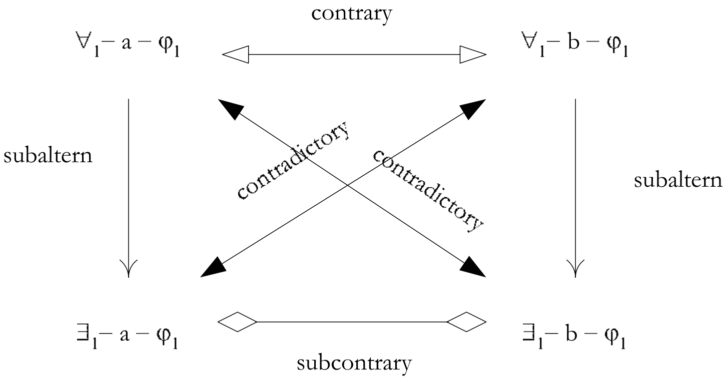

Figure 5.2: Logical square

Contradictory elementary poles can be identified on the basis of a simple syntactical criterion:

contradictory poles: A pole A and a pole B are *contradictory* if B can be generated from A by replacing the innermost a-pole / b-pole by the innermost b-pole / a-pole and by replacing every universal / existential quantifier by an existential / universal quantifier.

It suffices to identify a *sufficient* condition to identify *one* contradictory pole of A, as we can define any other contradictory pole on the basis of identifying equivalent poles. However, in elementary predicate logic, any elementary pole has only one contradictory pole.

According to the definition of contradictory poles, a contradictory pole can be constructed directly for each pole. In the following, the pole contradictory to the pole A will be denoted by $\overline{A}$. In predicate logic, a *wff* $\neg A$ is contradictory to a *wff* A. The definition of contradictory elementary poles can be traced back to the common definition of contradictory *wff*s by translating elementary poles into elementary closed structures (ecs). Given the ecs A, its negation, $\neg A$, is converted to an ecs with the converse sequence of quantifiers. This can be accomplished by applying definitions of quantifiers and double negation elimination (DN) negation. The resulting ecs contains a negation sign preceding the propositional function iff the ecs A does not contain a negation sign preceding the propositional function. Thus, the translation of the contradictory pole of pole A from the negation of the corresponding ecs A is derived by applying equivalence rules.

Contradictory pole-groups are already given by the construction of the a- and b-pole-groups of the same *wff* A. Analogously to the definition of contradictory poles, contradictory pole-groups are also definable, according to the following rule:

contradictory pole-groups: Pole-groups A and pole-groups B are *contradictory* if B can be generated from A by including the contradictory poles to each pole in A in different pole-groups in B iff their counter-parts are in the same pole-group in A.

This follows from converting the negation of a $\bigvee\bigwedge ecs$ to a $\bigvee\bigwedge ecs$ by applying DMG and DIS 1 before applying definitions of quantifiers and DN.

Contrary and subcontrary poles / pole-groups are definable by relying on contradictory and subaltern poles / pole-groups. Instead of the traditional term "subaltern," one may also speak of implication: B is subaltern to A iff A implies B. If A implies B, we will abbreviate this by "$A \vdash B$". If A is contrary to B, we will write "$A \lhd\!-\!\rhd B$," and if A and B are subcontrary "$A \diamond\!-\!\diamond B$". The following definitions are valid for any poles or pole-groups and the respective *wff*.

contrariety: $A \lhd\!-\!\rhd B$ iff $B \vdash \overline{A}$.

subcontrariety: $A \diamond\!-\!\diamond B$ iff $\overline{A} \vdash B$.

As $A \dashv\vdash A$ (and $\overline{A} \dashv\vdash \overline{A}$), any contradictory poles / pole-groups / *wff*s are also contrary as well as subcontrary poles / pole-groups / *wff*s.

Thus, the internal relations can be traced back to the internal relations of contradiction and implication. Contradictory poles and pole-groups can be generated directly from any pole or pole-groups. Thus, it only remains to define a general rule to identify the relations of implication between elementary poles and pole-groups. As the relation of implication is transitive, it suffices to define rules of implication such that B can be derived their iterative application. The main idea of the ab-notation developed in this book is to define the relation of implication between poles and, in general, between pole-groups, by iteratively applying derivation rules *in terms of operations* and reducing the identification of truth conditions of formulae hereunto.

In the following, we first limit our explanations to implications between elementary poles before considering implications between elementary pole-groups. In the framework of elementary predicate logic, it can be presumed that poles with different predicate letters are logically independent, and poles with opposed innermost poles a and b do not imply each other. Thus, in the realm of elementary predicate logic, the relation of implication between poles can be reduced to internal relations between their prefixes. In the following, we define the relations of implication between elementary poles, first for monadic and then for polyadic elementary predicate logic.

5.3.2 Monadic Elementary Predicate Logic

The relations of implications between elementary poles are determined by the following rule:

$$\forall_1 \vdash t_1 \vdash \exists_1.$$

This means that a pole with prefix $\forall_1$ implies which pole one receives by replacing $\forall_1$ by t_1. A pole with prefix t_1 implies the pole that one receives by replacing t_1 by $\exists_1$. These rules correspond to applying universal elimination and existential introduction to the corresponding ecs. Due to the property of transitivity, it also follows that poles with the prefix $\forall_1$ imply those poles that one receives by replacing $\forall_1$ by $\exists_1$. By applying the definitions of contradictory, contrary, and subcontrary poles, the complete internal relations of poles of elementary monadic predicate logic can be constructed, cf. figure 5.3. To accomplish this, it must be kept in mind that contradictory relations are also contrary and subcontrary. In monadic elementary predicate logic, the only poles in existence are of the form represented in figure 5.3. This figure comprises all internal relations of monadic elementary predicate logic.

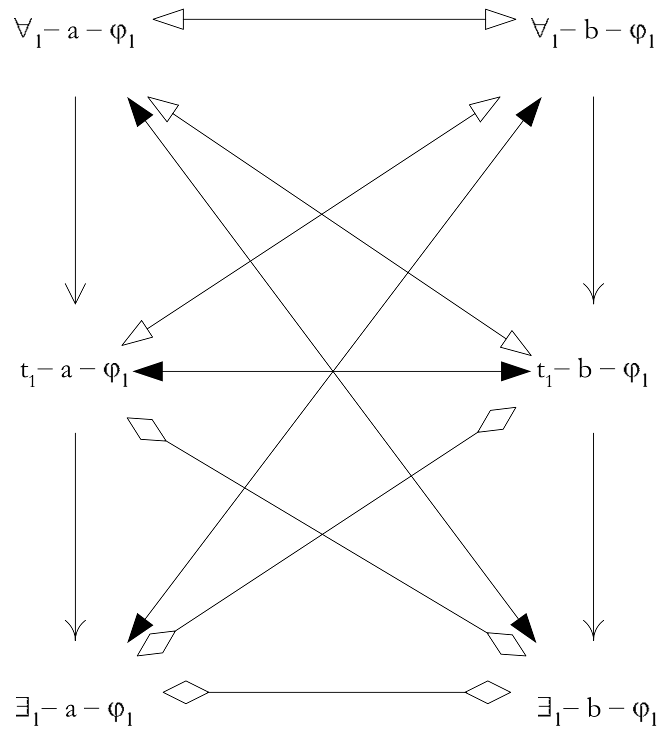

Figure 5.3: Internal relations between poles of monadic elementary predicate logic

The correctness and completeness of this schema can be proven by translating the poles in ecs and applying known definitions and derivation rules. Alternatively, the relations of implication can be proven semantically by referring to paraphrases: If *all* objects satisfy a certain propositional function, a *certain* object does as well. And if a *certain* object satisfies a propositional function, then *at least one* does. The completeness of the schema can be proven by constructing counter-models for all other possible relations of poles and their translations into *wff*s. This, again, can be made clear by their paraphrases: If *some* object satisfies a propositional function, this does not mean that a *certain* object satisfies this propositional function, or that *all* objects satisfy it. If a *certain* object satisfies a propositional function, this does not mean that *all* objects satisfy the propositional function. Furthermore, it is plain that if some / a certain / all objects satisfy a propositional function, this does not mean that some / a certain / all objects do *not* satisfy the same propositional function or that they satisfy some *other* propositional function. Hence, correctness and completeness can be demonstrated by simply referring to the correct understanding of the notation. No other proof can guarantee a higher degree of certainty or deeper insight.

The crucial point is that the relations of poles can be reduced to a finite number of symbolizing properties. These symbolizing properties can be generated systematically and, for every single transition of a symbolizing property to another symbolizing property, it can be decided whether or not a relation of implication is justified. The relations of implication between poles, therefore, make up a system that can be specified by rules defining the transitions between symbolizing properties. According to Wittgenstein's terminology, this amounts to a definition of a system in terms of operations. The main issue is that a logical problem, which involves identifying truth conditions of *wff*s of elementary predicate logic, is reduced to a problem solvable by specifying operations.

Symbolizing properties of propositional poles are solely the type (shape) of propositional variables and the innermost poles. In monadic elementary predicate logic, the kind of relation to the argument position of a propositional function is a further symbolizing property. The argument position can be related to either a universal or an existential quantifier, or to a name. $\forall_1 \vdash t_1$ and $t_1 \vdash \exists_1$ define the possible truth conserving syntactical transitions between these symbolizing properties. All the other possible transitions between this finite number of symbolizing properties are not truth conserving, as can easily been seen in their paraphrase.

Contrary to monadic elementary predicate logic, two further symbolizing properties are added in polyadic elementary predicate logic: (i) the order of quantifiers, (ii) closed forks. The relations of implication between these symbolizing properties are captured by the seven rules of implication in table 5.6. In addition to universal quantifier elimination and existential quantifier introduction, the quantifier exchange rule and the two rules for introducing and eliminating closed forks, respectively, are introduced. These rules will be explained in the following section, and then a procedure will be specified to completely define the relations of implication of elementary poles by iteratively applying these seven rules.

5.3.3 Polyadic Elementary Predicate Logic

5.3.3.1 Rules of Implication

$\exists\forall Ex$:	$\exists\mu\forall\nu \vdash \forall\nu\exists\mu$		
$\forall E$:	$\forall\mu \vdash t\mu$	$\exists I$:	$t < \genfrac{}{}{0pt}{}{\mu}{\nu} \vdash t\mu, \exists\nu$
$< I1$:	$\forall\mu, \forall\nu \vdash \forall < \genfrac{}{}{0pt}{}{\mu}{\nu}$	$< E1$:	$\exists < \genfrac{}{}{0pt}{}{\mu}{\nu} \vdash \exists\mu, \exists\nu$
$< I2$:	$\exists\mu\forall\nu \vdash \exists < \genfrac{}{}{0pt}{}{\mu}{\nu}$	$< E2$:	$\forall < \genfrac{}{}{0pt}{}{\mu}{\nu} \vdash \forall\mu\exists\nu$

Table 5.6: Rules of implication

Each rule is paraphrased in the following. Subsequent to paraphrases of the rules, we explain assumptions of their definitions.

PARAPHRASE:

$\exists\forall$-exchange rule ($\exists\forall Ex$): A pole A implies a pole B, if B is generated from A by exchanging the order of an existential quantifier $\exists\mu$ and a universal quantifier $\forall\nu$, $\forall\nu$ directly succeeding $\exists\mu$:

$$\exists\mu\forall\nu \vdash \forall\nu\exists\mu$$

universal quantifier elimination ($\forall E$): A pole A implies a pole B, if B is generated from A by eliminating a universal quantifier preceding the sequence of quantifiers and replacing it by a name:

$$\forall\mu \vdash t\mu$$

existential quantifier introduction ($\exists I$): A pole A implies a pole B, if B is generated from A by eliminating an arbitrary number i ($i > 0$) of prongs of a name's fork. These prongs make up the fork of a new existential quantifier preceding the sequence of quantifiers in B:

$$t < \genfrac{}{}{0pt}{}{\mu}{\nu} \vdash t\mu, \exists\nu$$

fork introduction 1 ($< I1$): A pole A implies a pole B, if B is generated from A by replacing two universal quantifiers of a sequence of universal quantifiers by one universal quantifier, and connecting the numbers of the column μ and ν by a fork:

$$\forall\mu, \forall\nu \vdash \forall < \begin{matrix}\mu \\ \nu\end{matrix}$$

fork-introduction 2 ($< I2$): A pole A implies a pole B, if B is generated from A by eliminating a universal quantifier $\forall\nu$ and connecting its column of numbers with the column of numbers of an existential quantifier $\exists\mu$ by a closed fork. $\exists\mu$ directly precedes $\forall\nu$ in A and is the only element of a sequence of existential quantifiers:

$$\exists\mu\forall\nu \vdash \exists < \begin{matrix}\mu \\ \nu\end{matrix}$$

fork-elimination 1 ($< E1$): A pole A implies a pole B, if B is generated from A by eliminating an arbitrary number i ($i > 0$) of the prongs of the fork of an existential quantifier $\exists < {}^{\mu}_{\nu}$. Those prongs make up the fork of a new existential quantifier in B. The resulting two existential quantifiers replace $\exists < {}^{\mu}_{\nu}$:

$$\exists < \begin{matrix}\mu \\ \nu\end{matrix} \vdash \exists\mu, \exists\nu$$

fork-elimination 2 ($< E2$): A pole A implies a pole B, if B is generated from A by eliminating an arbitrary number i ($i > 0$) of the prongs of a fork of a universal quantifier $\forall < {}^{\mu}_{\nu}$. Those prongs make up the fork of a new existential quantifier in B. $\forall < {}^{\mu}_{\nu}$ precedes a sequence of universal quantifiers in A. The resulting two quantifiers replace $\forall < {}^{\mu}_{\nu}$:

$$\forall < \begin{matrix}\mu \\ \nu\end{matrix} \vdash \forall\mu\exists\nu$$

GENERAL EXPLANATIONS:

- The rules specify only those symbolizing properties that vary left and right of $\vdash$.
- μ, ν are variables of columns of numbers. Thus, "$<{}^{\mu}_{\nu}$" means that an arbitrary number of numbers is connected by a fork. As a limiting case, the fork has only one prong such that $<{}^{\mu}_{\nu} = \mu$.
- If the whole fork following a name is eliminated by applying $\exists I$, then the name is eliminated too.

- It is tacitly presumed that, after applying $\exists\forall Ex$, $\forall E$, $\exists I$, and $< E2$, the *quantifier ordering rule*, cf. p. 162, and the *name rule*, cf. p. 162, are applied, if possible.
- Regarding $\forall E$, it is stipulated that, when replacing a universal quantifier by a name occurring in A, applying the *name rule* also implies that the two occurrences of the same name with different forks are merged to one name with one fork: $t\mu, t\nu \Rightarrow t <^{\mu}_{\nu}$.[2]
- As the order of those elements being separated by commas in the prefix is not significant, every such element can be treated as the first or last element. In particular, the following holds: A universal quantifier $\forall\nu$ precedes / succeeds an existential quantifier $\exists\mu$ "directly" iff the sequence of universal quantifiers containing $\forall\nu$ directly precedes / succeeds the sequence of existential quantifiers containing $\exists\mu$, and vice versa.
    - $\exists\forall Ex$, therefore, is applicable to each existential quantifier $\exists\mu$ and to each universal quantifier $\forall\nu$ of a sequence of existential quantifiers directly preceding a sequence of universal quantifiers. $\forall\nu\exists\mu$ directly follows the remaining existential quantifiers of that sequence of existential quantifiers and comes directly before the remaining universal quantifiers of that sequence of universal quantifiers.
- The rules are not only valid for the preceding quantifier of the sequence of quantifiers, but to every quantifier of the sequence of quantifiers.
- However, the rules are defined such that they only specify *direct* relations of implication. In particular, this holds for the application of $\exists\forall Ex$; the remaining rules are defined such that applications of $\exists\forall Ex$ are not missed. For this reason, it is referred to a universal quantifier, $\forall <^{\mu}_{\nu}$, preceding a sequence of universal quantifiers when applying $< E2$. By replacing $\forall <^{\mu}_{\nu}$ with $\forall\mu\exists\nu$, the new existential quantifier is as far as possible to the left of universal quantifiers. This then makes it possible to apply $\exists\forall Ex$ to its maximal extent after having applied $< E2$. The same holds for $\exists I$ by virtue of the stipulation that the prefix begins with the enumeration of names. $< I2$ refers to an existential quantifier directly preceding a universal quantifier, and that is the only element of a sequence of existential quantifiers. This is because by applying $\exists\forall Ex$, any universal quantifier can be brought forward before applying $< I2$. For the same reason, $\forall E$ applies only to universal

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[2] Alternatively, one could define $\forall E$ in terms of $\forall\mu \vdash t\mu$ in case of replacing the universal quantifier by a new name and in terms of $t\mu, \forall\nu \vdash t <^{\mu}_{\nu}$ in case of replacing the universal quantifier by a name occurring in A.

quantifiers that precede the sequence of quantifiers.

- When applying $< E2$ and $< I2$, not only forks are eliminated or introduced, but universal quantifiers are eliminated as well. $\forall E$ introduces forks if the universal quantifier is replaced by a name occurring in A. $\exists I$ eliminates forks if not all prongs of a name's fork are assigned to a new existential quantifier. In this respect, the rule's names are, to some extent, arbitrary.

5.3.4 Generating the Relations of Implication

In this section, we define a general procedure for generating the relations of implication between elementary poles.

All of the relations of implication between elementary poles can be constructed systematically by iteratively applying the seven rules defined in the preceding section. The "strongest poles", i.e. those poles not resulting from any application of the seven rules, make up the starting point of the construction. These poles are $\forall$-poles:

$\forall$-pole: $\forall$-poles are poles of elementary logic consisting of a sequence of n universal quantifiers preceding the inner pole, a or b. These, in turn, precede a predicate letter φ that is followed by numbers 1 to n. As a limiting case, we also call elementary poles of propositional logic "$\forall$-poles" with $n = 0$. We abbreviate $\forall$-poles as follows:

$$\forall_1, \ldots \forall_n - a/b - \varphi_{1 \ldots n}$$

Whereas $\forall$-poles are the starting points of the rules of implication, $\exists$-poles are the final results of their application.

$\exists$-pole: $\exists$-poles are poles of elementary logic consisting of a sequence of n existential quantifiers preceding the inner pole, a or b. These, in turn precede a predicate letter φ that is followed by numbers 1 to n. As a limiting case, we also call elementary poles of propositional logic "$\exists$-poles" with $n = 0$. We abbreviate $\exists$-poles as follows:

$$\exists_1, \ldots \exists_n - a/b - \varphi_{1 \ldots n}$$

starting rule (S-R.): $\forall$-poles are the initial poles of the application of the rules of implication.

The prefixes of initial poles are defined inductively as follows: (i) $\forall_1$ is a prefix of an initial pole, (ii) if $\forall_1, \ldots, \forall_i$ is a prefix of an initial pole, then $\forall_1, \ldots, \forall_{i+1}$ is a prefix of an initial pole, too, and (iii) prefixes of initial poles are only those prefixes set up by (i) or (ii). According to this inductive definition, an infinite number of initial poles can be generated. However, the following is valid: (i) No initial pole implies another initial pole. This is because of the following facts: (α) two different initial poles with identical prefix differ either by the predicate letter or the innermost pole, and (β) initial poles with different prefixes differ by the number of argument positions of the propositional functions. (ii) Poles implied by an initial pole are not implied by any other initial pole. The reason for this is that all seven rules of implication only vary the prefix of poles.[3] Because of these two facts (i) and (ii), every initial elementary pole constitutes the head of a closed system of relations of implication between elementary poles. As the rules of implication only concern the prefixes, the construction can be limited to the construction of prefix systems. The prefixes of the initial poles are the initial prefixes. Thus, the starting points of prefix systems can be introduced by S-R. The initial prefixes define the arity of a respective prefix system. For example, $\forall_1$ is the head of the monadic elementary prefix system that is defined completely by the application of $\forall E$ and $\exists I$, starting from $\forall_1$. Each single prefix system is defined by the arity of its initial pole. In the following, we define how one generates prefix systems of elementary predicate logic of arbitrary arity. However, one further restriction must be considered. Because of the definition of $\forall E$, the universal quantifier can be replaced by any name. As the number of names is unlimited, according to the alphabet of Q, the definition of $\forall E$ does not limit the replacement of universal quantifiers to some finite number of names. Thus, in order to specify an arbitrary, but finite, prefix system, we confine it to a certain number n ($n \geq 1$) of names. Thus, we define m-ary prefix systems allowing for n names.

The construction of such prefix systems can be based upon the following rule:

construction rule: Starting from an initial prefix of the form $\forall_1, \ldots \forall_n$ $(n \geq 0)$, all possible applications of the seven implication rules are to be carried out iteratively if the conditions of their application are satisfied.

The "conditions of application" of the seven rules are defined in the antecedent of each rule. In case of $\forall E$, it must be considered that the universal quantifier precedes the sequence of quantifiers. For $< I2$, it must be considered that the existential quantifier is the only element of a sequence of existential quantifiers.

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[3] We presume the completeness of the seven rules as demonstrated below, section 5.3.5.2.

"All possible applications" of the seven rules of a m-ary prefix system allowing for n names result from considering (i) all syntactical features satisfying the condition of each rule and (ii) all possible variants of applying the respective rule. In the following we specify how many times the respective rules should be applied for any given prefix.

$\exists\forall Ex$: $\exists\forall Ex$ must be applied to each universal quantifier of each sequence of universal quantifiers that directly follows a sequence of existential quantifiers. Given a sequence of universal quantifiers with k $(1 \leq k < m)$ universal quantifiers that directly follows a sequence of existential quantifiers with i existential quantifiers $(1 \leq i < m)$, then $\exists\forall Ex$ must be applied $k \times i$ times to the universal quantifiers of this sequence of universal quantifiers. Given l such sequences of universal quantifiers in the prefix, the number of universal quantifiers of the j-th sequence of universal quantifiers is k_j, and the number of the directly preceding j-th sequence of existential quantifiers is i_j, then $\exists\forall Ex$ must be applied

$$\sum_{j=1}^{j=l} k_j \times i_j$$

times to the respective poles.

$\forall E$: Each universal quantifier of the sequence of universal quantifiers preceding the sequence of quantifiers in the prefix must be replaced by all n names. Given that the number of universal quantifiers of that sequence is i $(i \leq m)$, then $\forall E$ must be applied

$$i \times n$$

times to the respective poles.

$\exists I$: Any number, r $(1 \leq r \leq m)$, of a fork's prongs of every name must be eliminated. Given the fork with name t in the prefix has i prongs $(1 \leq i \leq m)$, then $\exists I$ is applicable $2^i - 1$-times in order to eliminate the prongs of t's fork. Given k names and that the number of prongs of the j-th name's fork is i_j, then $\exists I$ must be applied

$$\sum_{j=1}^{j=k} 2^{i_j} - 1$$

times to the respective poles.

$< I1$: $< I1$ is applicable to all possible pairs of universal quantifiers of each sequence of universal quantifiers. Given that the number of universal quantifiers of a sequence of universal quantifiers is i ($2 \leq i \leq m$), then $< I1$ is $\sum\limits_{l=2}^{l=i}(l-1)$-times applicable to the universal quantifiers of this sequence of universal quantifiers. Given k sequences of universal quantifiers ($1 \leq k \leq \frac{m}{2}$) and that the respective number of the universal quantifiers of the j-th sequence of universal quantifiers is i_j ($2 \leq i_j \leq m$), then $< I1$ must be applied

$$\sum_{j=1}^{j=k}\sum_{l=2}^{l=i_j}(l-1)$$

times to the respective poles.

$< I2$: $< I2$ is applicable to each universal quantifier of a sequence of universal quantifiers that directly follow an existential quantifier, which is the only element of a sequence of existential quantifiers. Given a sequence of universal quantifiers with i universal quantifiers, then $< I2$ is i-times applicable to the universal quantifiers of this sequence of universal quantifiers. In addition, given k ($1 \leq k < \frac{m}{2}$) sequences of universal quantifiers in the prefix that directly follow an existential quantifier, which is the only element of a sequence of existential quantifiers, and given that the number of universal quantifiers of the j-th sequence of universal quantifiers is i_j, then $< I2$ must be applied

$$\sum_{j=1}^{j=k} i_j$$

times to the respective poles.

$< E1$: Forks of any existential quantifier with i ($1 < i \leq m$) prongs are eliminated such that two new existential quantifiers result. Their forks have at least 1 and less then i prongs. Given the number of a fork's prongs of an existential quantifier $\exists\mu$ is i ($1 < i \leq m$), then $< E1$ is $\frac{2^i-2}{2}$-times applicable to $\exists\mu$. In addition, given k existential quantifiers and that the number of the fork's prongs of the j-th existential quantifier is i_j, then $< E1$ must be applied

$$\sum_{j=1}^{j=k}\frac{2^{i_j}-2}{2}$$

times to the respective poles.

$< E2$: Forks of any universal quantifier with i $(1 < i \leq m)$ prongs are eliminated such that universal quantifiers result with forks having at least 1 and less then i prongs. Given that the number of a fork's prong of a universal quantifier $\forall\nu$ is i $(1 < i < m)$, then $< E2$ is $2^i - 2$-times applicable to $\forall\nu$. In addition, given k universal quantifiers and that the number of the fork's prongs of the j-th universal quantifier is i_j, then $< E2$ must be applied

$$\sum_{j=1}^{j=k} 2^{i_j} - 2$$

times to the respective poles.

Figure 5.4, p. 179 exemplifies the application of the *construction rule* for a binary prefix system with two names. This prefix system also demonstrates that none of the seven implication rules is dispensable. Each of the seven implication rules justifies a relation of implication that no other rule justifies.

5.3.5 Correctness and Completeness

This section demonstrates the correctness and completeness of the seven implication rules for identifying the relations of implications between elementary poles. A pole A implies a pole B iff a pole A implies a pole B according to our 7 rules of implication. In short:

$$A \vdash B \text{ iff } A \vdash_7 B$$

The direction from right to left expresses correctness of our rules; if $A \vdash_7 B$ then $A \vdash B$. This is proven syntactically by reducing these rules to common rules of predicate logic, and semantically by paraphrasing the respective poles according to the semantics of the ab-notation. The completeness of our seven rules is maintained by the direction from left to right; if $A \vdash B$ then $A \vdash_7 B$. This, we prove on the basis that all possible syntactic variations between poles that are not captured by our seven rules of implication are invalid. We reduce those invalid transitions to 9 syntactic variations and prove their invalidity semantically by paraphrasing them. Thus, if it is not possible to derive a pole B from a pole A by our 7 rules of implication, any transition from A to B must depend on one of

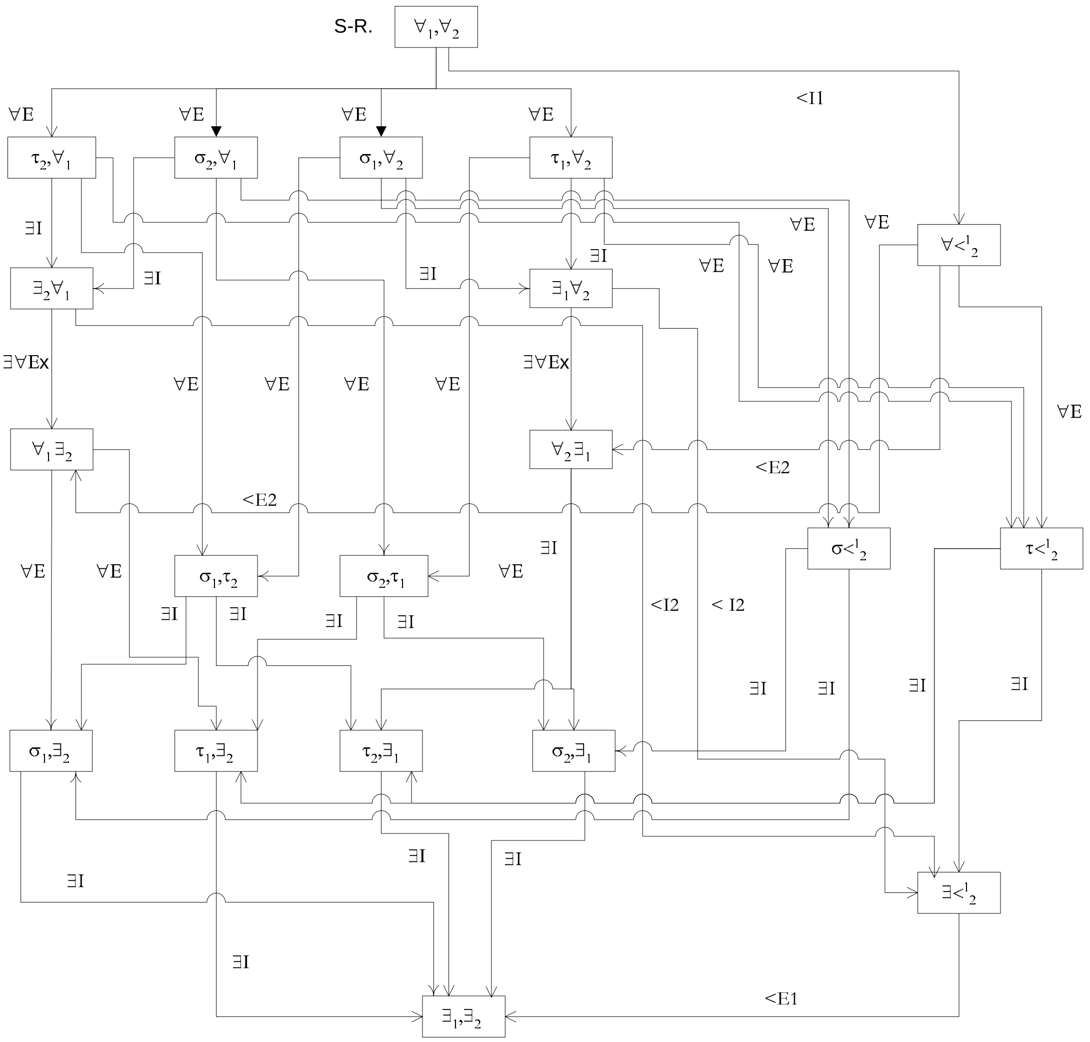

Figure 5.4: prefix system of binary, elementary predicate logic with two names

the nine invalid syntactic variation. In other words: if $A \nvdash_7 B$ then $A \nvdash B$, which is the contraposition from what is to be proven.

Before we go on to explain our "proofs" of correctness and completeness in detail, we want to point out that within New Logic a "proof" in a strict sense is a formal proof. The only acceptable criteria to which a proof refers are syntactic criteria, cf. the conception of a "logical proof" explained on p. 118. Proofs of correctness and completeness do not satisfy these criteria. They argue about proofs of a formal system and not within it. According to Wittgenstein, they are part of the "prosa" accompanying the real proofs. The function of this prosa is to explain the proof procedure and to convince the reader that it is intelligible and fulfills its purpose. If one calls this a "proof" this is meant in the lax sense of an "informal reasoning". According to Wittgenstein's point of view, confusing proof and prosa is one reason why it is overlooked that so called "metalogical proofs" are not immune against fallacies based on misunderstanding the logic of language, e.g. the misconception of speaking about formal properties as if they were material ones, cf. section 3.1. Using formalizations or structuring the informal argumentation in a logical manner is not substitute for a proof in a strict sense because in this context the adequacy of the used formalization and the correctness of the argumentation is not beyond question. We emphasize the distinction between proofs in a strict, formal sense and proofs in the lax sense of an informal reasoning by avoiding semi-formal expressions and by abstaining from the distinction between "Lemmata", "Theorems" and "Proofs".

5.3.5.1 Correctness

The correctness is proven syntactically by translating the rules in meta-formulae of *wff*s and by specifying a derivation scheme in a predicate calculus proving the respective *wff*s. This presupposes some predicate calculus. Given the Gentzen-Lemmon calculus of predicate logic (GLK$_Q$[4]) the implication rules can be based upon trivial successions of the universal quantifier elimination ($\forall E$), universal quantifier introduction ($\forall I$), existential quantifier introduction ($\exists I$), and existential quantifier elimination ($\exists E$) of GLK$_Q$. In contrast to the seven rules of implication, the rules of GLK$_Q$ can only be applied to the outermost quantifier. However, as the translations of elementary poles, the ecs, are in NNF (negative normal

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[4] cf. Lemmon (1978) and Lampert (2005a). Referring to the GLK$_Q$ by proving the correctness, of course, does not mean that the seven rules of implication are of the same kind as the rules of GLK$_Q$. Instead, unlike the rules of the GLK$_Q$, they do not refer to assumptions and other derivations but only to variations of one symbolic property.

form), outermost quantifiers can be eliminated first and reintroduced by $\forall E$, $\forall I$, $\exists I$, or $\exists E$ after the derivation corresponding to the respective rule of implication is completed. Thus, regarding the proof of correctness, we can confine ourselves to specifying the derivation scheme corresponding to the respective rule of implication. This is done in table 5.7.

name	**rule**	**translation**	**GLK$_Q$-derivation**	**comment**
$\exists\forall Ex$	$\exists\mu\forall\nu \vdash \forall\nu\exists\mu$	$\exists\mu\forall\nu A(\mu,\nu) \vdash \forall\nu\exists\mu A(\mu,\nu)$	A, $\forall E$, $\exists I$, $\exists E$, $\forall I$	A introduces the auxiliary assumption preparing $\exists E$.
$\forall E$	$\forall\mu \vdash t\mu$	$\forall\mu A(\mu) \vdash A(\mu/t)$	$\forall E$	
$\exists I$	$t < {}^{\mu}_{\nu} \vdash t\mu,\exists\nu$	$A(t) \vdash \exists\nu A(t/\nu)$	$\exists I$	
$< I1$	$\forall\mu, \forall\nu \vdash \forall < {}^{\mu}_{\nu}$	$\forall\mu\forall\nu A(\mu,\nu) \vdash \forall\mu A(\nu/\mu)$	$\forall E$, $\forall E$, $\forall I$	μ and ν are to be replaced by identical names.
$< I2$	$\exists\mu\forall\nu \vdash \exists < {}^{\mu}_{\nu}$	$\exists\mu\forall\nu A(\mu,\nu) \vdash \exists\mu A(\nu/\mu)$	A, $\forall E$, $\exists I$, $\exists E$	A introduces the auxiliary assumption preparing $\exists E$; $\forall E$ replaces ν by the name introduced by A; this name is replaced by μ in virtue of $\exists I$.
$< E1$	$\exists < {}^{\mu}_{\nu} \vdash \exists\mu, \exists\nu$	$\exists\mu A(\mu) \vdash \exists\mu\exists\nu A(\mu/\nu)$	A, $\exists I$, $\exists I$, $\exists E$	A introduces the auxiliary assumption preparing $\exists E$; applying $\exists I$ twice replaces occurrences of the same name introduced by A by different variables.

name	**rule**	**translation**	**GLK$_Q$-derivation**	**comment**
$< E2$	$\forall <^{\mu}_{\nu} \vdash \forall\mu\exists\nu$	$\forall\mu A(\mu) \vdash$ $\forall\mu\exists\nu A(\mu/\nu)$	$\forall E, \exists I, \forall I$	$\forall E$ replaces μ by t; $\exists I$ replaces some but not all occurrences of t by ν; $\forall I$ replaces the remaining occurrences of t by μ.

Table 5.7: GLK$_Q$-Derivation of the rules of implication

However, the proof of correctness does not depend on any other system of logic because it can also be carried out by merely referring to the paraphrase of poles. This is done by proving the rules semantically. As explained in 5.2.1, the poles identify combinations of objects that must or must not occur in $\Im(\varphi)$ in order to satisfy the conditions of the truth and falsehood of formulae.

SEMANTIC JUSTIFICATION OF THE RULES OF IMPLICATION:

$\exists\forall Ex$**:** If *some* object at argument positions μ combined with *all* objects at argument positions ν satisfies (does not satisfy) $\Im(\varphi)$, then *all* objects at argument positions ν combined with *some* object at argument positions μ satisfy (do not satisfy) $\Im(\varphi)$.

$\forall E$**:** If *all* objects at argument positions μ satisfy (do not satisfy) $\Im(\varphi)$, then a *certain* object at argument positions μ satisfies (does not satisfy) $\Im(\varphi)$.

$\exists I$**:** If a *certain* object at argument positions ν satisfies (does not satisfy) $\Im(\varphi)$, then *some* object at argument positions ν satisfies (does not satisfy) $\Im(\varphi)$.

$< I1$**:** If *all* objects at argument positions μ combined with *all* objects at argument positions ν satisfy (do not satisfy) $\Im(\varphi)$, then *all* objects, the *same* at argument positions μ and ν, satisfy (do not satisfy) $\Im(\varphi)$.

$< I2$**:** If an object at argument positions μ combined with *all* objects at argument positions ν satisfies (does not satisfy) $\Im(\varphi)$, then some object, the *same* at argument positions μ and ν, satisfies (does not satisfy) $\Im(\varphi)$.

$< E1$: If an object, the *same* at argument positions μ and ν, satisfies (does not satisfy) $\Im(\varphi)$, then an object at argument positions μ combined with *some* object at argument positions ν satisfies (does not satisfy) $\Im(\varphi)$.

$< E2$: If all objects, the *same* at argument positions μ and ν, satisfy (do not satisfy) $\Im(\varphi)$, then all objects at argument positions μ combined with *some* object at argument positions ν satisfy (do not satisfy) $\Im(\varphi)$.

5.3.5.2 Completeness

The following proves that the seven rules of implication specify the relations of implication between the elementary poles completely. This is done by scrutinizing all possible minimal variations of the pole's symbolizing properties one by one, and proving that any variations that is not justified by one of the seven rules is not truth preserving. The symbolizing properties of two poles vary minimally if the two poles are identical except for one symbolizing property. Thus, for example, the pole $\exists <{}^{1}_{2}\forall_3 - a - F_{123}$ and the pole $\forall_3\exists <{}^{1}_{2} - a - F_{123}$ vary minimally, because they only vary by the ordering of two quantifiers. In contrast, $\exists <{}^{1}_{2}\forall_3 - a - F_{123}$ and $\forall_3\exists_1\exists_2 - a - F_{123}$ do not vary minimally because, in addition to a variation of quantifiers, they differ with regard to the connection of numbers by prongs of a fork. For variations in the connection of numbers by the prongs of forks, one must consider that a *minimal* variation exists if two numbers are connected in pole A but not in pole B, or vice versa. This is irrespective of any variations concerning the relations of the respective argument positions to names or quantifiers. Thus, for example, $\forall <{}^{1}_{2} - a - F_{12}$ and $\forall_1\exists_2 - a - F_{12}$ vary minimally because the numbers 1 and 2 are connected by the prongs of fork in the former pole but not in the latter pole. The (finite) totality of all possible minimal variations of symbolizing properties of two elementary poles is enumerated in table 5.9.

To prove that the seven rules of implication specify the relations of implication of all elementary poles completely, it suffices to prove that they specify the relations of implication between poles that vary *minimally*. This is because every further variation of symbolizing properties either does not constitute a relation of implication or leads to a weaker pole. From the latter, it follows that the initial pole a forteriori also implies this weaker pole. Hence, the rules of implication identify only the slightest possible weakenings of the pole's truth conditions that are expressible by elementary predicate logic.

The minimal variations not constituting a relation of implication can be reduced to the nine rules listed in table 5.8. We refer to these rules as “$\overline{\text{rules of implication}}$”. While the seven rules of implication specify the minimal weakenings of the pole’s truth conditions, the nine $\overline{\text{rules of implication}}$ identify the minimal variations of the pole’s truth conditions that do not constitute relations of implication.

$\overline{\varphi/\psi}$:	$\phi \nvdash \psi$	$\overline{a/b}$:	$a-\varphi \nvdash b-\varphi$ $b-\varphi \nvdash a-\varphi$
$\overline{\forall\exists Ex}$:	$\forall\mu\exists\nu \nvdash \exists\nu\forall\mu$		
$\overline{tI}$:	$s\mu \nvdash t\mu$	$\overline{\exists E}$:	$\exists\mu \nvdash t\mu$
$\overline{<E1}$:	$\forall <^{\mu}_{\nu} \nvdash s\mu, t\nu$	$\overline{<I1}$:	$s\mu, t\nu \nvdash \exists <^{\mu}_{\nu}$
$\overline{<E2}$:	$\forall <^{\mu}_{\nu} \nvdash \exists\mu\forall\nu$	$\overline{<I2}$:	$\forall\mu\exists\nu \nvdash \exists <^{\mu}_{\nu}$

Table 5.8: $\overline{\text{rules of implication}}$

PARAPHRASE:

$\overline{\varphi/\psi}$: A pole A does not imply a pole B, if A and B do not contain the same propositional function:

$$\phi \nvdash \psi$$

$\overline{a/b}$: A pole A does not imply a pole B if A and B do not contain the same innermost pole:

$$a-\varphi \nvdash b-\varphi$$
$$b-\varphi \nvdash a-\varphi$$

$\overline{\forall\exists Ex}$: A pole A does not imply a pole B if $\forall\mu$ precedes $\exists\nu$ in A but succeeds $\exists\nu$ in B:

$$\forall\mu\exists\nu \nvdash \exists\nu\forall\mu$$

$\overline{tI}$: A pole A does not imply a pole B if the name s refers to an argument position μ in A, whereas in B the name t ($s \neq t$) refers to μ:

$$s\mu \nvdash t\mu$$

$\overline{\exists E}$: A pole A does not imply a pole B if an existential quantifier refers to an argument position μ in A, whereas in B a name refers to μ:

$$\exists\mu \not\vdash t\mu$$

$\overline{< E1}$: A pole A does not imply a pole B if one and the same universal quantifier refers to argument positions μ and ν in A, whereas in B different names refer to μ and ν:

$$\forall < \begin{array}{c}\mu\\ \nu\end{array} \not\vdash s\mu, t\nu$$

$\overline{< E2}$: A pole A does not imply a pole B if one and the same universal quantifier refers to argument positions μ and ν in A, whereas in B an existential quantifier refers to μ preceding an universal quantifier referring to ν:

$$\forall < \begin{array}{c}\mu\\ \nu\end{array} \not\vdash \exists\mu\forall\nu$$

$\overline{< I1}$: A pole A does not imply a pole B if two different names refer to argument positions μ and ν in A, whereas in B one and the same existential quantifier refers to μ and ν:

$$s\mu, t\nu \not\vdash \exists < \begin{array}{c}\mu\\ \nu\end{array}$$

$\overline{< I2}$: A pole A does not imply a pole B if a universal quantifier referring to the argument positions μ precedes an existential quantifier referring to the argument positions ν in A, whereas in B an existential quantifier refers to μ and ν:

$$\forall\mu\exists\nu \not\vdash \exists < \begin{array}{c}\mu\\ \nu\end{array}$$

GENERAL EXPLANATIONS:

- All the variations of symbolizing properties captured by the nine rules are sufficient to exclude relations of implication. It is impossible for the pole A to imply pole B if one of the nine rules must be applied to pass over from A to B.
- The rules also hold if the variables for columns of numbers are connected with other variables of numbers by a fork. Thus, for example, $\overline{\exists E}$ is also valid in case of $\exists < {}^{\mu}_{\nu} \nvdash t\mu, \exists\nu$.
- Contrary to the rules of implication, it is not presumed that the quantifiers succeed one another directly.

In analogy with the rules of implication, these rules can be proven semantically by referring to the paraphrase of poles.

SEMANTICAL JUSTIFICATION OF THE $\overline{\text{RULES OF IMPLICATION}}$:

$\overline{\varphi/\psi}$: If some combinations of objects satisfy (do not satisfy) $\Im(\varphi)$, then these combinations of objects do not *eo ipso* satisfy (not satisfy) $\Im(\psi)$.

$\overline{a/b}$: If some combinations of objects *satisfy* $\Im(\varphi)$, then these combinations of objects do not *eo ipso not* satisfy $\Im(\varphi)$ and vice versa.

$\overline{tI}$: If a certain object s satisfies (does not satisfy) $\Im(\varphi)$ at an argument position μ, then *another* object t does not *eo ipso* satisfy (not satisfy) $\Im(\varphi)$ at μ.

$\overline{\exists E}$: If *some* object satisfies (does not satisfy) $\Im(\varphi)$ at argument position μ , then a *certain* object t does not *eo ipso* (not) satisfy $\Im(\varphi)$ at μ.

$\overline{< E1}$: If all objects, the *same* at argument positions μ and ν, satisfy (do not satisfy) $\Im(\varphi)$, then a certain object s does not *eo ipso* (not) satisfy $\Im(\varphi)$ at argument positions μ in combination with *another* certain object t at argument positions ν.

$\overline{< E2}$: If all objects, the *same* at argument positions μ and ν, satisfy (do not satisfy) $\Im(\varphi)$, then some object does not *eo ipso* (not) satisfy $\Im(\varphi)$ at argument positions μ in combination with all objects at argument positions ν.

$\overline{< I1}$: If a certain object s satisfies (does not satisfy) $\Im(\varphi)$ at argument positions μ in combination with *another* certain object t at argument positions ν, then some object, the *same* at argument positions μ and ν, does not *eo ipso* (not) satisfy $\Im(\varphi)$.

$\overline{< I2}$**:** If all objects satisfy (do not satisfy) $\Im(\varphi)$ at argument positions μ in combination with some object at argument positions ν, then some object, the *same* at argument positions μ and ν, does not *eo ipso* (not) satisfy $\Im(\varphi)$.

To reduce all possible minimal variations of symbolizing properties to the seven rules of implication and the nine $\overline{\text{rules of implication}}$, and in order to differentiate minimal variations justifying a relation of implication and minimal variations not justifying a relation of implication, two further rules must be considered:

Weakening of the antecedent (WA): If the transition of a symbolizing property X to a symbolizing property Y is not justified, then the transition of a symbolizing property X*, derived by an application of one of the seven rules of implication to X *a forteriori* is also not justified.

Strengthening of the consequent (SC): If the transition of a symbolizing property X to a symbolizing property Y is not justified, then the transition of X to the symbolizing property Y*, from which Y can be derived by applying one of the seven implication rules, *a forteriori* is also not justified.

These rules follow from the definition of implication and the correctness of the seven rules of implication. In fact, we will only make use of the two rules of implication, $\forall E$ and $\exists I$, when applying WA or SC. WA and SC complement the rule of transitivity (T) regarding implication: if a transition of a symbolizing property X to a symbolizing property Y is justified, and if the transition of Y to a symbolizing property Z is justified, then the transition from X to Z is *a forteriori* justified. WA and SC articulate corresponding rules for transitions of symbolizing properties that do not justify implications.

Table 5.9 lists all possible syntactical minimal variations of symbolizing properties of elementary poles. Two poles differ by at least one of the following properties: (i) predicate letter, (ii) the innermost pole, (iii) the ordering of quantifiers, (iv) the fact that one and the same argument position is not related in both cases to an universal quantifier / existential quantifier / the same name, or (v) two numbers of argument positions are connected by the prongs of a fork in one pole but not in the other. All these variations are reduced either to the rules of implication or to the $\overline{\text{rules of implication}}$ in table 5.9. In case of the latter, WA and SC are applied if necessary. This is denoted in the last column of table 5.9: "a.f. l.7: SC, $\forall E$" in line 8 means, for example, that it follows *a forteriori* from line 7 by applying SC and $\forall E$ that the minimal variation listed in line 8 is not justified. It is not presumed that the mentioned applications of rules in the last columns are the only possible ones to reduce the variations to some other cases. The variation in line 11 ($\exists\mu \Rightarrow \forall\mu$), for example, can be reduced just as well by WA and $\exists I$ to line 8. For the proof of completeness, it suffices to mention *one* possible application of rules that excludes a relation of implication. The seven rules of implication are noted simply to demonstrate that they comprise all cases not reducible to rules of implications. A star (*) in lines 13, 16, 33, 35 subsumes all further possible minimal variations. It is used when all corresponding variations are reducible to preceding cases referred to in the last column. Thus, for example, “$* \Rightarrow \forall <^{\mu}_{\nu}$ l.8,11” in the second column of line 13 means that all further minimal variations of pole A regarding the symbolizing property $\forall <^{\mu}_{\nu}$ of pole B are reducible to lines 8 and 11.

no.	**variation**	**rule**
1.	$\varphi \Rightarrow \psi$	$\overline{\varphi/\psi}$
2.	$a - \varphi \Rightarrow b - \varphi$ $b - \varphi \Rightarrow a - \varphi$	$\overline{a/bEx}$
3.	$\exists\mu\forall\nu \Rightarrow \forall\nu\exists\mu$	$\exists\forall Ex$
4.	$\forall\nu\exists\mu \Rightarrow \exists\mu\forall\nu$	$\overline{\forall\exists Ex}$
5.	$\forall\mu \Rightarrow t\mu$	$\forall E$
6.	$\forall\mu \Rightarrow \exists\mu$	$\forall E, \exists I$
7.	$s\mu \Rightarrow t\mu$	$\overline{tI}$
8.	$t\mu \Rightarrow \forall\mu$	a.f. l.7: SC, $\forall E$
9.	$t\mu \Rightarrow \exists\mu$	$\exists I$
10.	$\exists\mu \Rightarrow t\mu$	$\overline{\exists E}$
11.	$\exists\mu \Rightarrow \forall\mu$	a.f. l.10: SC, $\forall E$
12.	$\forall\mu, \forall\nu \Rightarrow \forall <^{\mu}_{\nu}$	$< I1$
13.	$* \Rightarrow \forall <^{\mu}_{\nu}$	l.8,11
14.	$\forall\mu, t\nu \Rightarrow t <^{\mu}_{\nu}$	$\forall E$

no.	**variation**	**rule**
15.	$\forall\mu, \forall\nu \Rightarrow t <{}^{\mu}_{\nu}$	$< I1, \forall E$
16.	$* \Rightarrow t <{}^{\mu}_{\nu}$	l.7,10
17.	$\exists\mu\forall\nu \Rightarrow \exists <{}^{\mu}_{\nu}$	$< I2$
18.	$t\mu\forall\nu \Rightarrow \exists <{}^{\mu}_{\nu}$	$\exists I, \exists\forall Ex, < I2$
19.	$\forall\mu\forall\nu \Rightarrow \exists <{}^{\mu}_{\nu}$	$\forall E, \exists I, \exists\forall Ex, < I2$
20.	$\forall\mu\exists\nu \Rightarrow \exists <{}^{\mu}_{\nu}$	$\overline{< E2}$
21.	$s\mu, t\nu \Rightarrow \exists <{}^{\mu}_{\nu}$	$\overline{< I1}$
22.	$t\mu\exists\nu \Rightarrow \exists <{}^{\mu}_{\nu}$	a.f. l.21: WA, $\exists I$
23.	$\exists\mu\exists\nu \Rightarrow \exists <{}^{\mu}_{\nu}$	a.f. l.22: WA, $\exists I$
24.	$\forall <{}^{\mu}_{\nu} \Rightarrow t\mu, s\nu$	$\overline{< E1}$
25.	$\forall <{}^{\mu}_{\nu} \Rightarrow \forall\mu, t\nu$	a.f. l.24: SC, $\forall E$
26.	$\forall <{}^{\mu}_{\nu} \Rightarrow \forall\mu,\forall\nu$	a.f. l.25: SC, $\forall E$
27.	$\forall <{}^{\mu}_{\nu} \Rightarrow \exists\mu\forall\nu$	$\overline{< E2}$
28.	$\forall <{}^{\mu}_{\nu} \Rightarrow \forall\mu\exists\nu$	$< E2$
29.	$\forall <{}^{\mu}_{\nu} \Rightarrow t\mu, \exists\nu$	$< E2, \forall E$
30.	$\forall <{}^{\mu}_{\nu} \Rightarrow \exists\mu, \exists\nu$	$< E2, \forall E, \exists I$
31.	$t <{}^{\mu}_{\nu} \Rightarrow t\mu, \exists\nu$	$\exists I$
32.	$t <{}^{\mu}_{\nu} \Rightarrow \exists\mu, \exists\nu$	$\exists I, \exists I$
33.	$t <{}^{\mu}_{\nu} \Rightarrow *$	l.7,8
34.	$\exists <{}^{\mu}_{\nu} \Rightarrow \exists\mu, \exists\nu$	$< E1$
35.	$\exists <{}^{\mu}_{\nu} \Rightarrow *$	l.10,11

Table 5.9: Reduction of all minimal variations to the 7 rules of implication and the 9 $\overline{\text{rules of implication}}$

The seven rules of implication specify the relations of implications between elementary poles correctly and completely because they comprise those, and only those, minimal variations of symbolizing properties that justify a relation of implication.

5.3.6 Decidability

This section specifies an effective procedure for deciding whether a given elementary pole A implies a given elementary pole B. We will assume this procedure in the solution of the equivalence problem specified in the next section.

If B differs from A by a symbolizing property reducible to one of the 9 rules of implication, it can be concluded that A does not imply B. For example, the following pole A

$$\forall_1\exists_3\forall_2\forall_4 - b - F_{1234}$$

does not imply pole B

$$t_4,\exists <\begin{smallmatrix}1\\2\\3\end{smallmatrix} - b - F_{1234}.$$

This is because $< E2$ must be applied, due to the connection of the numbers 2 and 3 by a fork in B. However, the contrary is not valid. If a pole A and a pole B do not differ in any symbolizing property reducible to one of the 9 rules of implication, it cannot be concluded that A implies B. For example, the pole A

$$\exists_1\forall < \begin{smallmatrix}2\\3\end{smallmatrix} - a - F_{123}$$

does not imply the pole B

$$\forall_3\exists < \begin{smallmatrix}1\\2\end{smallmatrix} - a - F_{123}.$$

The symbolizing differences of these two poles can be analysed in the following three minimal variations: $\exists_1\forall_3 \Rightarrow \forall_3\exists_1$, $\exists_1\forall_2 \Rightarrow \exists < \begin{smallmatrix}1\\2\end{smallmatrix}$ and $\forall < \begin{smallmatrix}2\\3\end{smallmatrix} \Rightarrow \forall_3\exists_2$. None of these variations can be reduced to one of the 9 rules of implication. Instead, any of these variations by themselves are justified by a rule of implication, namely by $\exists\forall Ex$, $< I2$ and $< E2$. However, B cannot be derived from A by successively applying the seven rules of implications. On the contrary, as soon as one of the three rules $\exists\forall Ex$, $< I2$ or $< E2$ is applied to A, one of the nine $\overline{\text{rules of implication}}$ must be applied to that result in order to pass to B.

If $\overline{\varphi/\psi}$ and $\overline{a/b}$ are not applicable whether A implies B solely depends on their prefixes. Thus, to decide in this case whether the elementary pole A implies the elementary pole B, it suffices to construct the m-ary prefix system of n names. In this case, m is the number of arguments of the propositional function contained in A and B. Meanwhile n is the number of names contained in A and B, that is at most $2 \times m$. If no name occurs in A and no name occurs in B, then the name a must be introduced arbitrarily by $\forall E$ to guarantee the application of $\exists I$. Consequently, the following holds for n: $1 \leq n \leq 2 \times m$. Therefore, the construction of the prefix system terminates. Thus, it can be decided whether A implies B by relying on this prefix system. Such systems can also be laid down as databases that are consulted if required.

In the following, we will specify another efficient procedure to decide whether a pole A implies a pole B. On the one hand, we will make use of the fact that the $\overline{\text{rules of implication}}$ define sufficient conditions for excluding a relation of implication. As soon as a transition from A to B is only possible by a $\overline{\text{rule of implication}}$, the procedure stops. On the other hand, the symbolizing properties of A shall be varied systematically, such that B is derived from A if possible. To achieve this, we will make use of necessary conditions to derive B from A. If A and B differ by the fact that numbers of argument positions are not connected by a fork in A that are connected by a fork in B, or if A and B differ by the fact that in A numbers of argument positions are connected by a fork that are not connected by a fork in B, then applying the following rules is a necessary condition to derive B from A:

- $< I1$ to connect two forks that are related to universal quantifiers,
- $< I2$ to connect two forks, one related to an existential quantified and the other to a universal quantifier,
- $\forall E$ to connect two forks, one related to a name and the other to a universal quantifier,
- $< E2$ to disconnect numbers of argument positions in the case of universal quantifiers[5],

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[5] Numbers of argument positions not related to an existential quantifier in B must remain related to a universal quantifier subsequent to the application of $< E2$, if possible. Thus, for example, given that A is $\forall < {}^{1}_{2} - a - F_{12}$ and B is $\forall_1\exists_2 - a - F_{12}$, then $< E2$ must be applied such that B and not $\forall_2\exists_1 - a - F_{12}$ is the result.

- $\exists I$ to disconnect numbers of argument positions in the case of names[6],
- $< E1$ to disconnect numbers of argument positions in the case of existential quantifiers.

Therefore, insofar as relations of implications are not excluded by a $\overline{\text{rule of implication}}$, forks can be introduced or eliminated by applying the respective rules $< I1$,$< I2$, $\forall E$, $< E2$, $\exists I$, $< E1$ until numbers of argument positions in A are connected by prongs iff they are connected by prongs in B.[7] Finally, the relation of argument positions to quantifiers / names, and also the order of quantifiers, is to be adjusted by applying $\forall E$, $\exists I$ and $\exists\forall Ex$, if possible. Furthermore, one can make use of the following facts:

- It is only necessary to test once whether $\overline{\varphi\psi SUB}$ or $\overline{a/b}$ are applicable, because none of the rules of implication changes predicate letters or innermost poles.
- Given that the process of adjusting the connection of numbers by forks in A and B is completed successfully, it is not necessary to test whether the resulting pole and the pole B differ in their connections by forks. This is because applying $\forall E$ and $\exists I$ to adjust the relations of argument positions to quantifiers / names, and applying $\exists\forall Ex$ to adjust the ordering of quantifiers, will not change the connections by forks any more.
- Due to the definition of $\forall E$ and $\exists I$, new existential quantifiers precede the sequence of quantifiers in any case. Therefore, $A \vdash B$ can be derived if (i) A and B do not differ in their connections of the numbers of argument positions by forks, and if (ii) $\overline{tI}$, $\overline{\exists E}$, or $\overline{\forall\exists Ex}$ do not have to be applied to pass from A to B.

Given these points, the following practicable procedure can decide whether a given elementary pole A implies a given elementary pole B:

Decision rule for elementary poles ($\mathcal{D}$-R eP):

1. Test whether $\overline{\varphi/\psi}$, $\overline{a/b}$, $\overline{\forall\exists Ex}$, $\overline{tI}$, $\overline{\exists E}$, $\overline{< E1}$, $\overline{< E2}$, $\overline{< I1}$, or $\overline{< I2}$ must be applied in order to pass to B. If so $A \nvdash B$.

---

[6] Numbers of argument positions not related to an existential quantifier in B must remain related to names after applying $\exists I$, if possible.

[7] If necessary applying these rules must be prepared by $\exists\forall Ex$.

2. Apply, proceeding from the inside to the outside successively, $< I1$, $< I2$, and $\forall E$ to connect those numbers of argument positions by a fork in A that are connected by a fork in B. If necessary, prepare the application of the rules by $\exists\forall Ex$. If, at any stage, $\overline{< I1}$ or $\overline{< I2}$ must be applied to connect numbers of argument positions that are connected by a fork in B, by a fork in A, too, then $A \nvdash B$.

3. Test whether $\overline{\forall\exists Ex}$, $\overline{tI}$, $\overline{\exists E}$, $\overline{< E1}$, or $\overline{< E2}$ must be applied to pass to B. If so, then $A \nvdash B$.

4. Apply, proceeding from the inside to the outside successively, $< E1$, $< E2$, and $\exists E$ to eliminate forks connecting numbers of argument positions in A that are not connected by a fork in B. If necessary, prepare the application of the rules by $\exists\forall Ex$. If, at any stage, $\overline{< E1}$ or $\overline{< E2}$ must be applied to eliminate a fork connecting numbers of argument positions in A that are not connected by a fork in B, then $A \nvdash B$.

5. Test whether $\overline{\forall\exists Ex}$, $\overline{tI}$, or $\overline{\exists E}$ must be applied to pass to B. If so, then $A \nvdash B$. If not, then $A \vdash B$.

According to $\mathcal{D}$-R eP, derivation schemata can be constructed starting with A. Every derivation step applies one of the seven rules of implication or one of the nine $\overline{\text{rules of implication}}$ to the pole of the preceding line. These schemata will end by applying one of the nine $\overline{\text{rules of implication}}$ in the case of $A \nvdash B$. Meanwhile, in the case of $A \vdash B$, they terminate with a pole that does not differ from B regarding the connection of numbers of argument positions by forks. By applying $\forall E$, $\exists I$, and $\exists\forall Ex$, the pole B can be derived from this pole in the latter case. For completeness sake, this shall be done in the following derivation schemata. Thus, in the case of $A \vdash B$, the derivation schemata will only make use of the seven rules of implication in the manner specified by $\mathcal{D}$ and end up with B.

EXAMPLE 1:

$$\exists_4\forall_5\exists_3\forall < {}^1_2 - b - F_{12345} \quad \nvdash \quad \forall_1\exists_4\forall_5\exists < {}^2_3 - b - F_{12345}$$

no.	**pole**	**rule**
1.	$\exists_4\forall_5\exists_3\forall < {}^1_2 - b - F_{12345}$	A

no.	**pole**	**rule**
2.	$\exists_4\forall_5\exists <^{1}_{\substack{2\\3}} - b - F_{12345}$	$< E1$
3.	$t_1, \exists_4\forall_5\exists < {}^{2}_{3} - b - F_{12345}$	$\overline{\exists E}$

EXAMPLE 2:

$$t_1, \exists < {}^{8}_{9}\forall_4, \forall_5\exists < {}^{6}_{7}\forall < {}^{2}_{3} - a - F_{123456789}$$

$$\vdash$$

$$t_3, \forall < {}^{4}_{5}\exists < {}^{1}_{2}, \exists_6, \exists_7, \exists_8, \exists_9 - a - F_{123456789}$$

no.	**pole**	**rule**
1.	$t_1, \exists < {}^{8}_{9}\forall_4, \forall_5\exists < {}^{6}_{7}\forall < {}^{2}_{3} - a - F_{123456789}$	A
2.	$t_1, \exists < {}^{8}_{9}\forall < {}^{4}_{5}\exists < {}^{6}_{7}\forall < {}^{2}_{3} - a - F_{123456789}$	$< I1$
3.	$t <^{1}_{\substack{2\\3}}, \exists < {}^{8}_{9}\forall < {}^{4}_{5}\exists < {}^{6}_{7} - a - F_{123456789}$	$\forall E$
4.	$t <^{1}_{\substack{2\\3}}, \exists < {}^{8}_{9}\forall < {}^{4}_{5}\exists_6, \exists_7 - a - F_{123456789}$	$< E1$
5.	$t <^{1}_{\substack{2\\3}}, \exists_8, \exists_9\forall < {}^{4}_{5}\exists_6, \exists_7 - a - F_{123456789}$	$< E1$
6.	$t_3, \exists < {}^{1}_{2}, \exists_8, \exists_9\forall < {}^{4}_{5}\exists_6, \exists_7 - a - F_{123456789}$	$\exists I$
7.	$t_3, \forall < {}^{4}_{5}\exists < {}^{1}_{2}, \exists_6, \exists_7, \exists_8, \exists_9 - a - F_{123456789}$	$\exists\forall Ex$

5.3.7 System of implication between elementary pole-groups

In this section, we define how to generate the whole system of implications between pole-groups of elementary logic. We proceed analogously to the case of the system of implications between poles.

5.3.7.1 Rules

To generate the whole system of implications between elementary pole-groups, we must extend the rules of implication between poles. We further add rules for introducing and eliminating poles and pole-groups. In contrast to the 7 prefix rules, we call these rules "suffix rules".

P E:	$\{A,B\} \vdash \{B\}$	PG I:	$\{B\} \vdash \{B\}, \{C\}$
P I:	$\{A,B\} \vdash \{A,A,B\}$	PG E:	$\{B\}, \{B\} \vdash \{B\}$
$\top$ I:	$\{B\} \vdash \{B,A\},\{B,\overline{A}\}$	$\perp$ E:	$\{B\},\{A,\overline{A}\} \vdash \{B\}$

Table 5.13: Rules of implication

The rules are defined for pole-groups either being all a- or all b-pole-groups. A is a meta-variable for one pole, B,C are meta-variables for an arbitrary number of poles.

PARAPHRASE:

pole elimination (P E): A pole-group $\{A, B\}$ implies a pole-group $\{B\}$, that is, the pole-group that is generated from $\{A, B\}$, by eliminating the pole A. It suffices to limit P E to the elimination of an $\exists$-pole:

$$\{B, \exists_1, \ldots \exists_n - a/b - \varphi_{1 \ldots n}\} \vdash \{B\}$$

pole introduction (P I): A pole-group $\{A,B\}$ implies a pole-group $\{A,A,B\}$, that is, the pole-group that is generated from $\{A,B\}$, by introducing the pole A once more:

$$\{A,B\} \vdash \{A,A,B\}$$

tautology introduction ($\top$ I)[8]: A pole-group $\{B\}$ implies the enumeration of pole-groups $\{B,A\}, \{B,\overline{A}\}$, that is, the pole-groups generated from $\{B\}$, by adding A and $\overline{A}$:

$$\{B\} \vdash \{B,A\},\{B,\overline{A}\}$$

pole-group introduction (PG I): A pole-group $\{B\}$ implies the enumeration of pole-groups $\{B\}, \{C\}$, that is, the pole-groups generated from $\{B\}$, by adding $\{C\}$. It suffices to limit PG I to the introduction of pole-groups containing $\forall$-poles:

$$\{B\} \vdash \{B\}, \{\forall_1, \ldots \forall_n - a/b - \varphi_{1 \ldots n}, \ldots\}$$

pole-group elimination (PG E): The pole-groups $\{B\}, \{B\}$ imply the pole-group $\{B\}$, that is, the pole-groups generated from $\{B\}, \{B\}$, by eliminating $\{B\}$:

$$\{B\}, \{B\} \vdash \{B\}$$

contradiction elimination ($\perp$ E): The enumeration of pole-groups $\{B\}, \{A,\overline{A}\}$ implies the single pole-group $\{B\}$, that is, the pole-group generated from $\{B\}$,$\{A,\overline{A}\}$, by eliminating $\{A,\overline{A}\}$:

$$\{B\},\{A,\overline{A}\} \vdash \{B\}$$

5.3.7.2 Correctness

The correctness of these rules follows from the fact that they are special cases of corresponding rules of predicate logic:

name	**rule**	**translation**
P E	$\{A,B\} \vdash \{B\}$	$A \wedge B \vdash B$
P I	$\{A,B\} \vdash \{A,A,B\}$	$A \wedge B \vdash A \wedge A \wedge B$
$\top$I	$\{B\} \vdash \{B,A\},\{B,\overline{A}\}$	$B \vdash B \wedge A \vee B \wedge \neg A$
PG I	$\{B\} \vdash \{B\}, \{C\}$	$B \vdash B \vee C$
PG E	$\{B\}, \{B\} \vdash \{B\}$	$B \vee B \vdash B$
$\perp$ E:	$\{B\},\{A,\overline{A}\} \vdash \{B\}$	$B \vee A \wedge \neg A \vdash B$

Note that P I, $\top$ I, PG E, $\perp$ E correspond to rules that are also valid for the direction from right to left. However, this direction can be reduced to the application of P E and P I. All 6 rules can be applied to any conjunct or disjunct of a $\bigvee\bigwedge ecs$.

Plainly, the correctness of the rules follows also semantically from the paraphrases of pole-groups. Poles specify necessary conditions of $\Im$; their paraphrases are combined by "and". From this the correctness of the first three rules follow:

P E: If some $\Im$ satisfies the conditions specified by the poles A and B, it satisfies the conditions specified by B.

P I: If some $\Im$ satisfies the conditions specified by the poles A and B, it satisfies the conditions by A and A and B.

---

[8] We call this rule *tautology introduction* as the corresponding rule in elementary predicate logic can be derived from introducing $(A \vee \neg A)$ by conjunction to the conjunction corresponding to the pole-group $\{B\}$ and then applying DIS 1.

$\top$ I: If some $\Im$ satisfies the conditions specified B, it either satisfies the conditions B and the condition A or the condition B and the condition contradicting A.

Pole-groups specify sufficient conditions of $\Im$; their paraphrases are combined by "or". From this the correctness of the latter three rules follow:

PG I: If some $\Im$ satisfies a condition B, it satisfies the condition B or the condition C.

PG E: If some $\Im$ either satisfies the condition B or the condition B, it satisfies the condition B.

$\bot$ E: If some $\Im$ satisfies a condition B or a contradictory condition, it satisfies the condition B.

5.3.7.3 Completeness

Contrary to the system of poles, the system of pole-groups is not dense. If we order pole-groups by the relation of implication, then there is always some further pole-group in between two other pole-groups. This is due to P I, $\top$ I and PG I that all allow one to introduce an arbitrary number of further poles. Given $\mathcal{B}$ can be derived by some rule from $\mathcal{A}$ it is always possible to derive $\mathcal{B}$ from $\mathcal{A}$ by intermediate applications of the rules involving P I, $\top$ I or PG I which introduce further poles. However, we define the rules such that implications are direct if one abstains from introducing further poles. That is why P E can be limited to $\exists$-poles as the weakest poles, and PG I can be limited to $\forall$-poles as the strongest poles. P E and PG E are limited to the elimination of only one pole or pole-group. $\bot$ E is limited to pole-groups containing only two contradictory poles because of P E. In addition, we limit P I and PG I to the introduction of only one pole or pole-group for the sake of not omitting other possible applications of P I or PG I.

Similar to prefix systems, systems of implications between pole-groups can be generated by starting from any pole-group only containing $\forall$-poles:

starting rule PG (S-R. PG): Pole-groups only containing $\forall$-poles are the initial pole-groups of the application of the 13 rules of implication.

general construction rule (GC-R.): Starting from any pole-group only containing $\forall$-poles, all possible applications of each of the 13 implication rules are to be carried out iteratively if the conditions of their application are satisfied.

The conditions of the application of the rules are defined by their antecedent. The possible applications of the seven rules concerning the prefix of the poles are already defined. If the number of the poles of all pole-groups is k, then each number of applications must be multiplied by k as each rule can be applied to each pole. The possible applications of the six further rules of implication are plain:

- Given that the number of $\exists$-poles of a pole-group is n, then P E can be applied n times to that pole-group. Given k pole-groups and the number of $\exists$-poles of the i-th ($1 \leq i \leq k$) pole-group is n_i, then P E can be applied

  $$\sum_{i=1}^{i=k} n_i$$

  times.
- Given that the number of poles of a pole-group is n, then P I can be applied n times. Given k pole-groups and the number of poles of the i-th ($1 \leq i \leq k$) pole-group is n_i, then P I can be applied

  $$\sum_{i=1}^{i=k} n_i$$

  times.
- Given identical pole-groups we limit the number of applications of PG E to these identical pole-groups to 1. The reason for this is that the order of the pole-groups is insignificant. Therefore, it makes no difference which one of identical pole-groups is eliminated. Thus, given n groups of identical pole-groups, then PG E can be applied n times.
- Given that the number of pole-groups with only two contradictory poles is n, then $\perp$ E can be applied n times.
- There are infinitely many possible applications of PG I, as an infinite number of $\forall$-poles and an infinite number of different pole-groups with $\forall$-poles can be generated.
- There are infinitely many possible applications of $\top$ I, as an infinite number of contradictory poles can be generated.

Although any rule can be applied in different ways, any application of one of the rules realizes only one possible application of all possible applications of a rule. For this reason any application of the 7 prefix rules derives only one pole from one pole. Thus, for example, $\forall E$ can either replace $\forall_1$ by a_1 or by b_1 but not both. Thus, limiting the substitution to these two names and applying $\forall E$ to

$$a - \{\forall_1 - a - F_1\} \tag{5.1}$$

results in one of the two following pole-groups:

$$a - \{a_1 - a - F_1\} \tag{5.2}$$
$$a - \{b_1 - a - F_1\}. \tag{5.3}$$

Yet, (5.1) also implies the following pole-group:

$$a - \{a_1 - a - F_1, b_1 - a - F_1\}. \tag{5.4}$$

However, applying $\forall E$ once to (5.1) cannot result in (5.4) because any step of derivation consists in only one possible application of the rules. Thus, by applying a single rule once we cannot realize all possible applications of that rule to one pole. Instead, we realize only one possible application. Notwithstanding, we are able to realize all different possible applications of the prefix rules from one and the same pole due to P I: Given some rule shall be applied m times in a different way, we can realize all these m applications from one and the same pole by introducing this pole $m-1$ times and then carry out all m possible applications of the rule iteratively. In case of deriving (5.4) from (5.1) PI must be applied once:

(1)	$a - \{\forall_1 - a - F_1\}$	(5.1)
(2)	$a - \{\forall_1 - a - F_1, \forall_1 - a - F_1\}$	P I
(3)	$a - \{a_1 - a - F_1, \forall_1 - a - F_1\}$	$\forall E$
(3)	$a - \{a_1 - a - F_1, b_1 - a - F_1\}$	$\forall E$

Thus, due to P I we can presume that all possible applications of the prefix rules to one pole are realized by the *general construction rule*. As P I is an equivalence rule, we make it possible to realize any arbitrary application of the prefix rules by equivalence transformation. In respect to the other 6 suffix rules there are no different possibilities to apply them to one pole or to pole-groups. Furthermore, by $\top I$ and $\vee I$ it is also possible to apply the rules to any poles and pole-groups not being elements of some set of pole-groups. Thus, the conditions of the application of the 13 rules can be satisfied to any arbitrary extent such that any possible application of the rules to any pole or pole-group is realized by the *general construction rule*. That is to say, we can presume the following:

GC-R.-Totality: By the *general construction rule* the totality of all possible derivations between elementary pole-groups according to our 13 rules of implication is generated.

Thus, the infinite totality of all possible derivations according to our calculus is defined by the totality of all possible iterative applications of the derivation rules. As these derivation rules have the form of operations, this definition satisfies Wittgenstein's criteria of defining an infinite totality. However, this does not amount to the completeness of our calculus as we have not yet shown that there might not be some valid derivation that is not captured by our derivations as generated by our 13 rules.

There is one important difference between the system of implications between elementary poles and elementary pole-groups. In case of relations between elementary poles any syntactic difference is a difference between symbolizing properties. That is to say, there is no possible equivalent transformation between symbolizing poles. The relation of implication is asymmetric as long as we consider only poles as relata. For this reason, certain syntactic differences between two poles are a sufficient condition to conclude that one pole does not imply the other. In contrast, in case of pole-groups syntactic differences may not symbolize. That is to say, two syntactically different sets of pole-groups may be equivalent. We call "*symbolizing* syntactic differences" syntactic differences that make a difference in the truth conditions of the formulae. We cannot conclude from certain syntactic differences that one set of pole-groups does not imply the other because they may not symbolize. Consider, for example, the following two pole-groups:

$$a - \{\forall_1 - a - F_1, \exists_1 - a - F_1\} \tag{5.5}$$

$$a - \{\forall_1 - a - F_1, a_1 - a - F_1\} \tag{5.6}$$

One can transit from (5.5) to (5.6) by applying $\overline{\exists E}$. However, from this it does not follow that (5.5) does not imply (5.6) because it might well be that the syntactic difference between those two pole-groups does not symbolize. There still might be some possible derivation of (5.6) from (5.5) by our 13 rules of implication. Thus, for example, (5.6) is derivable from (5.5) by our rules of implication:

(1)	$a - \{\forall_1 - a - F_1, \exists_1 - a - F_1\}$	(5.5)
(2)	$a - \{\forall_1 - a - F_1\}$	P E
(3)	$a - \{\forall_1 - a - F_1, \forall_1 - a - F_1\}$	P I
(4)	$a - \{\forall_1 - a - F_1, a_1 - a - F_1\}$	$\forall$ E

This derivation reveals that the second pole of the pole-group is irrelevant for the derivation of (5.6) from (5.5).

Equivalence relations are identified by the system of implications between pole-groups by the fact that the direction of implication is not directed in one way only. For example, (5.5) can in turn be derived from (5.6) by $\exists I$. We will make use of this fact by identifying equivalent pole-groups, for the sake of their minimization, in order to solve the equivalence problem in the next section.

However, by the *general construction rule* the totality of all possible derivations according to our 13 rules of implication is generated, cf. p. 200. Therefore, we can assume that the syntactic differences of a set of pole-groups $\mathcal{A}$ and a set of pole-groups $\mathcal{B}$ is irreducible to those syntactic differences specified by our 13 rules of implication if it is not generated by the *general construction rule*. In this case we can conclude that $\mathcal{B}$ cannot be implied by $\mathcal{A}$ because the validity of the syntactic transition from $\mathcal{A}$ to $\mathcal{B}$ must depend on one of the following three cases: (i) either the validity depends on some syntactic differences reducible to one of the nine rules of implications or, (ii) it depends on the introduction of a pole that cannot be derived by the 7 prefix poles from poles introduced by P I or $\top$ I or, (iii) it depends on the elimination of a pole-group that cannot be eliminated due to PG E or $\perp$ E and the prefix rules. In case of (i) we already showed in section 5.3.5.2 that such syntactic variations constitute invalid transitions if their application is necessary. From this it follows that in case (ii) a pole is introduced that is not implied by the consisting pole-groups, whereas in case (iii) a pole-group is eliminated that does not imply other pole-groups. In both cases this constitutes an invalid transitions as becomes clear by the respective paraphrases: If $\Im$ satisfies some condition A it does not *eo ipso* satisfy a condition not implied by A; if $\Im$ either satisfies some condition B or some condition C that is not implied by B it does not *eo ipso* satisfy C. Thus, in case $\mathcal{B}$ is not derivable from $\mathcal{A}$ according to our *general construction rule*, we can conclude that in this case a syntactic difference between $\mathcal{A}$ and $\mathcal{B}$ is a symbolizing difference such that some $\Im$ exists that is model of $\mathcal{A}$ and counter-model of $\mathcal{B}$:

$$\text{if} \mathcal{A} \nvdash_{13} \mathcal{B} \text{ then } \mathcal{A} \nvdash \mathcal{B}$$

This is the contraposition of maintaining completeness of our calculus. Thus, our calculus is not only correct but also complete.

Due to the completeness of our calculus the problem of implication is solved in elementary predicate logic by applying *the general construction rule*. The 13 rules of implication define a *calculus* satisfying Wittgenstein's standards. They all have the form of operations, and by their iterative application, any system of implications between elementary pole-groups can be generated from initial pole-groups introduced by S-R. PG. Introducing the initial pole-groups containing $\forall$-poles can itself be defined by operations systematically varying the number of universal quantifiers in the prefix, the inner pole, the use of predicate letters, and the number of poles. Thus, by means of a finite set of operations, the infinite possibilities of relations of implications are definable in the realm of elementary predicate logic.

5.3.7.4 Decidability

Similar to the decidability of implication between poles we can define either (i) a non-intelligent decision procedure by defining upper bounds of the application of the 13 rules of implication depending on the syntactic features of the respective sets of pole-groups $\mathcal{A}$ and $\mathcal{B}$ or (ii) some intelligent, effective procedure to decide upon $\mathcal{A} \vdash \mathcal{B}$.

(i) Any finite construction of a system of implications between pole-groups must be limited to a finite number of applications of $\forall$ E, P I, $\top$ I and PG I, and to a finite number of $\forall$-poles introduced by S-R. PG. To decide whether a set of pole-groups $\mathcal{A}$ implies another set of pole-groups $\mathcal{B}$, it obviously suffices to consider a system of implications that is limited by the number of names, poles, and pole-groups of these sets. Given k different names ($k \geq 0$), m different poles of both sets, n different pole-groups of the larger set, not more than $k + 1$ names (by $\forall$ E), not more than m $\forall$-poles (by S-R.PG) , m poles (by P I), m pairs of contradictory poles (by $\top$ I), and sets with no more than n pole-groups must be introduced. Thus, whether a given set of pole-groups implies another given set of pole-groups can be determined by generating a finite system of implications from these two sets.

(ii) In case of the system of implications between poles we defined an effective procedure to decide whether some pole A implies a pole B by identifying a derivation of B from A in case B is implied by A and by identifying some necessary invalid syntactic variation in case A does not imply B. However, we cannot proceed likewise to decide whether some pole-group $\mathcal{A}$ implies some pole-group $\mathcal{B}$ because of two reasons. First, not any syntactic variation is symbolizing. Thus, syntactic differences as those reducible to the nine rules of implication are not sufficient to conclude $\mathcal{A} \nvdash \mathcal{B}$. Second, the syntactic features of two pole-groups cannot be correlated as trivially as those of two poles. However, we specify a procedure that reduces the question whether a set of pole-groups $\mathcal{A}$ implies a set of pole-groups $\mathcal{B}$ to the decision upon the implication of poles. To do so, we introduce some terminology first.

contrary pole-groups: A pole-group A is contrary if two poles of A are contrary.

superfluous poles: Given two pole-groups $\mathcal{A}$ and $\mathcal{B}$ and two subcontrary poles, namely the pole A (contained in $\mathcal{A}$) and the pole B (contained in $\mathcal{B}$). Then A is superfluous if all the remaining poles of $\mathcal{B}$ are implied by the poles of $\mathcal{A}$.

derivative suffix pole-group: The derivative suffix pole-group of a pole-group A is the group of suffices one obtains by eliminating all prefixes of the poles in A.

correlated pole-groups: A pole-group of $\mathcal{A}$ (PG$_{\mathcal{A}}$) is correlated to a pole-group of $\mathcal{B}$ (PG$_{\mathcal{B}}$) if all suffices of the derivative suffix pole-group PG$_{\mathcal{B}}$ are contained in the derivative suffix pole-group of PG$_{\mathcal{A}}$.

Decision Rule for elementary pole-groups ($\mathcal{D}$-R ePG):

1. Eliminate contrary pole-groups in $\mathcal{A}$. If all pole-groups are eliminated, then $\mathcal{A} \vdash \mathcal{B}$.
2. Eliminate superfluous poles in $\mathcal{B}$ iteratively.[9] If all pole-groups are eliminated, then $\mathcal{A} \vdash \mathcal{B}$.
3. Correlate the remaining pole-groups $\mathcal{A}_r$ and $\mathcal{B}_r$ of $\mathcal{A}$ and $\mathcal{B}$: If not all remaining PG$_{\mathcal{A}_r}$ are correlated to some PG$_{\mathcal{B}_r}$, then $\mathcal{A} \nvdash \mathcal{B}$.

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[9] The exact process of eliminating superfluous poles is described in the following section 5.4.

4. Decide upon the implication of poles of correlated pole-groups: If some $\mathrm{PG}_{\mathcal{A}_r}$ exists such that not all poles of some correlated pole-group $\mathrm{PG}_{\mathcal{B}_r}$ are implied by the poles of $\mathrm{PG}_{\mathcal{A}_r}$, then $\mathcal{A} \nvdash \mathcal{B}$. Otherwise $\mathcal{A} \vdash \mathcal{B}$.

$\mathcal{D}$-R ePG 1 and 2 rest on equivalence transformations involving $\perp$ E and $\top$ I as is explained on p. 214f. Therefore, one can reduce the question whether a set of pole-groups implies another set of pole-groups to the remaining sets $\mathcal{A}_r$ and $\mathcal{B}_r$. If empty sets remain $\mathcal{A} \vdash \mathcal{B}$ follows because $\mathcal{A}$ is a contradiction and $\mathcal{B}$ a tautology in these cases. Plainly, if the poles of each pole-group of $\mathcal{A}_r$ imply all poles of some correlated pole-group of $\mathcal{B}_r$ $\mathcal{A}$ implies $\mathcal{B}$; in this case one can derive $\mathcal{B}_r$ from $\mathcal{A}_r$ by P I, the prefix rules, P E, PG I and PG E. However, the following also holds: if the poles of some pole-group of $\mathcal{A}_r$ do not imply all the poles of some correlated pole-group of $\mathcal{B}_r$, $\mathcal{A}_r \nvdash \mathcal{B}_r$ and therefore $\mathcal{A} \nvdash \mathcal{B}$. For in this case it is always possible to construct a counter-model, i.e. an $\Im$ that satisfies the conditions defined by the respective pole-group of $\mathcal{A}_r$ but does not satisfy the conditions specified by the pole-groups of $\mathcal{B}_r$ because it does not satisfy the condition defined by at least one pole of each pole-group of $\mathcal{B}_r$. Plainly, this is always possible if the criterion of $\mathcal{D}$-R ePG 4. is satisfied. Thus, one only has to consider the relations of implication for correlated pole-groups by considering relations between the respective poles.

EXAMPLE 1.

$$a - \{a - Q, b - Q\}, a - \{\forall < \tfrac{1}{2} - a - F_{12}\}, a - \{\exists < \tfrac{1}{2} - a - F_{12}, a - P\} \quad (5.7)$$

$$\vdash$$

$$a - \{\forall_1 \exists_2 - a - F_{12}, \exists_{1,2} - a - F_{12}, b - P\}, a - \{a - P\} \quad (5.8)$$

According to $\mathcal{D}$-R ePG 1 the first pole-group of (5.7) must be eliminated because $a - Q$ and $b - Q$ are contrary. According to $\mathcal{D}$-R ePG 2 $b - P$ in the first pole-group of (5.8) can be eliminated because it is superfluous. Thus, the following remains to be proven:

$$a - \{\forall < \tfrac{1}{2} - a - F_{12}\}, a - \{\exists < \tfrac{1}{2} - a - F_{12}, a - P\} \quad (5.9)$$

$$\vdash$$

$$a - \{\forall_1 \exists_2 - a - F_{12}, \exists_{1,2} - a - F_{12}\}, a - \{a - P\} \quad (5.10)$$

The derivative suffix pole-groups of the remaining pole-groups are the following: $a - \{a - F_{12}\}, a - \{a - F_{12}, a - P\}$ and $a - \{a - F_{12}, a - F_{12}\}, a - \{a - P\}$.

Thus, $a - \{\forall < \frac{1}{2} - a - F_{12}\}$ is correlated to $a - \{\forall_1\exists_2 - a - F_{12}, \exists_{1,2} - a - F_{12}\}$ and $a - \{\exists < \frac{1}{2} - a - F_{12}, a - P\}$ to $a - \{\forall_1\exists_2 - a - F_{12}, \exists_{1,2} - a - F_{12}\}$ as well as to $a - \{a - P\}$. Thus, all pole-groups of (5.9) are correlated to pole-groups of (5.10) and the criterion of $\mathcal{D}$-R 3 ePG cannot be applied. According to $\mathcal{D}$-R ePG 4 the relations of implication between the poles of correlated pole-groups must be considered. $\forall_1\exists_2 - a - F_{12}$ as well as $\exists_{1,2} - a - F_{12}$ are implied by $\forall < \frac{1}{2} - a - F_{12}$. $a - P$ is implied by $a - P$. Thus, there exists a correlated pole-group for all pole-groups of (5.9), such that all poles of that correlated pole-groups of (5.10) are implied by the poles of the respective pole-group of (5.9). Thus, the criterion in $\mathcal{D}$-R ePG 4 for $\mathcal{A} \not\vdash \mathcal{B}$ is not satisfied. Therefore $\mathcal{A} \vdash \mathcal{B}$.

One possible derivation according to the defined calculus is the following:

no.	**formula**	**rule**
(1)	$a - \{a - Q, b - Q\}, a - \{\forall < \frac{1}{2} - a - F_{12}\}, a - \{\exists < \frac{1}{2} - a - F_{12}, a - P\}$	(5.7)
(2)	$a - \{\forall < \frac{1}{2} - a - F_{12}\}, \{\exists < \frac{1}{2} - a - F_{12}, a - P\}$	$\perp$ E
(2)	$a - \{\forall < \frac{1}{2} - a - F_{12}, \forall < \frac{1}{2} - a - F_{12}\}, a - \{\exists < \frac{1}{2} - a - F_{12}, a - P\}$	P I
(3)	$a - \{\forall_1\exists_2 - a - F_{12}, \forall < \frac{1}{2} - a - F_{12}\}, a - \{\exists < \frac{1}{2} - a - F_{12}, a - P\}$	<E2
(4)	$a - \{\forall_1\exists_2 - a - F_{12}, a < \frac{1}{2} - a - F_{12}\}, a - \{\exists < \frac{1}{2} - a - F_{12}, a - P\}$	$\forall$E
(5)	$a - \{\forall_1\exists_2 - a - F_{12}, \exists < \frac{1}{2} - a - F_{12}\}, a - \{\exists < \frac{1}{2} - a - F_{12}, a - P\}$	$\exists$I
(6)	$a - \{\forall_1\exists_2 - a - F_{12}, \exists_{1,2} - a - F_{12}\}, a - \{\exists < \frac{1}{2} - a - F_{12}, a - P\}$	<E1
(7)	$a - \{\forall_1\exists_2 - a - F_{12}, \exists_{1,2} - a - F_{12}\}, a - \{a - P\}$	P E
(8)	$a - \{\forall_1\exists_2 - a - F_{12}, \exists_{1,2} - a - F_{12}, b - P\},$ $a - \{\forall_1\exists_2\frac{1}{2} - a - F_{12}, \exists_{1,2} - a - F_{12}, a - P\}, a - \{a - P\}$	$\top$ I
(9)	$a - \{\forall_1\exists_2 - a - F_{12}, \exists_{1,2} - a - F_{12}, b - P\},$ $a - \{\forall_1\exists_2\frac{1}{2} - a - F_{12}, a - P\}, a - \{a - P\}$	P E
(10)	$a - \{\forall_1\exists_2 - a - F_{12}, \exists_{1,2} - a - F_{12}, b - P\},$ $a - \{a - P\}, a - \{a - P\}$	P E
(11)	$a - \{\forall_1\exists_2 - a - F_{12}, \exists_{1,2} - a - F_{12}, b - P\}, a - \{a - P\}$	PG E

EXAMPLE 2.

$$a - \{a - Q, b - Q\}, a - \{\forall < \tfrac{1}{2} - a - F_{12}\}, a - \{\exists < \tfrac{1}{2} - a - F_{12}, a - P\} \tag{5.11}$$

$$\not\models$$

$$a - \{\exists_2\forall_1 - a - F_{12}, \exists_{1,2} - a - F_{12}, b - P\}, a - \{a - P\} \tag{5.12}$$

Application of $\mathcal{D}$-R ePG 1 to 3 are identical to EXAMPLE 1. However, $\forall < \frac{1}{2} - a - F_{12}$ does not imply $\exists_2\forall_1 - a - F_{12}$; one must apply $\overline{< E2}$ in this case. As $a - \{\forall < \frac{1}{2} - a - F_{12}\}$ is not correlated to any other pole-groups of (5.12), the criterion of $\mathcal{D}$-R ePG 4 is satisfied. Therefore $\mathcal{A} \not\models \mathcal{B}$.

5.4 Minimization

In this section, the equivalence problem (EP) is solved for elementary predicate logic by defining a procedure to convert symbolizing pole groups to ab-symbols. This is done by assuming the possibility of deciding implications between poles and pole-groups as described in the preceding sections. In fact, in elementary predicate logic, it suffices to refer to internal relations between poles to generate ab-symbols. In subsection 5.4.1, the equivalence problem is solved for propositional logic by referring to the Quine-McCluskey algorithm. In subsection 5.4.2, we then show how to generalize the Quine-McCluskey algorithm to solve the equivalence problem in the whole realm of elementary predicate logic.

5.4.1 Solution of the EP in propositional logic

Disjunctions of conjunctions in propositional logic are those special forms of $\bigvee\bigwedge ecs$ in which the elementary closed structures contain 0 quantifiers. In this section, we first describe the solution of the equivalence problem by referring to the reduced disjunctive normal forms of the Quine-McCluskey algorithm (5.4.1.1). We then show, in section 5.4.1.2, how to apply this solution to the pole-group notation. In subsection 5.4.2, we generalize this solution to solve the equivalence problem for $\bigvee\bigwedge ecs$ and elementary pole-groups in general.

5.4.1.1 Quine-McCluskey Algorithm

The equivalence problem is solvable by converting *wff*s to their canonical disjunctive normal forms (CDNF), and then generating the reduced disjunctive normal forms (RDNF) according to the Quine-McCluskey algorithm.

$$wff \Rightarrow CDNF \Rightarrow RDNF$$

The RDNF reduce the disjuncts of CDNF to their prime implicants, i.e. the minimized disjuncts. The RDNF contain *all* prime implicants. Starting with RDNF, the Quine-McCluskey algorithm searches at a second step for the *essential prime implicants*. Yet, contrary to the generation of RDNF, this search has no unique solution. To solve the equivalence problem, only the first step of the Quine-McCluskey algorithm, the construction of the RDNF, must be carried out.

To generate prime implicants, the single disjuncts are represented as classes. This is also necessary to solve the equivalence problem because, by representing disjuncts as classes, it is abstained from insignificant differences of the ordering of conjuncts and disjuncts. The disjuncts of CDNF are called "minterms". Minterms are minimized by applying the "merging rule" M-R:

$$\{A_1 \ldots A_n, B\}, \{A_1 \ldots A_n, \overline{B}\} \dashv\vdash \{A_1 \ldots A_n\} \quad M\text{-}R$$

This rule is based upon converting $(A_1 \wedge \ldots \wedge A_n \wedge B) \vee (A_1 \wedge \ldots \wedge A_n \wedge \neg B)$ to $A_1 \wedge \ldots \wedge A_n \wedge (B \vee \neg B)$ by DIS1, and then eliminating the tautologous conjunct $(B \vee \neg B)$. The direction from right to left of M-R corresponds to $\top$I. The direction from left to right of M-R corresponds to applying P E and PG E.

The RDNF are generated by recursively applying M-R, starting with the minterms of CDNF. These minterms are placed in a first table, the minterm table. By ordering the minterms in classes according to the number of negated terms, only a pair of minterms of neighbored classes must be compared to apply M-R. If M-R is applicable, the new term is placed in a further table. Compared pairs of terms must be marked, but the single terms can be used for further applications of M-R, until this rule cannot be applied any further to the terms of a table. Identical terms must be written down only once in a new table. This procedure is applied recursively until M-R is not applicable any further. All terms of tables not marked, to which M-R is not applicable any more, are prime implicants. The RDNF consists of the disjunction of all prime implicants.

The RDNF of a *wff* A is a *maximal disjunction* of minimal disjuncts. It enumerates *all* minimal sufficient conditions of the truth of A. The Quine-McCluskey algorithm generates, in a second step, *minimal disjunctions* of minimal disjuncts by minimizing RDNF. However, applying this step of the algorithm has no unique solution. The maximal disjunction of minimal disjuncts (5.13), for example, is equivalent to the two minimal disjunctions of minimal disjuncts (5.14) and (5.15):

$$P \wedge \neg Q \vee Q \wedge \neg P \vee P \wedge R \vee Q \wedge R \tag{5.13}$$

$$P \wedge \neg Q \vee Q \wedge \neg P \vee P \wedge R \tag{5.14}$$

$$P \wedge \neg Q \vee Q \wedge \neg P \vee Q \wedge R \tag{5.15}$$

Thus, the solution of the equivalence problem must abstain from minimizing disjunctions.

In contrast, by converting a *wff* A as well as its negation $\neg A$ to a CDNF and reducing the CDNFs to RDNFs, the equivalence problem is solved. By generating the CDNF of A as well as of $\neg A$, one yields 2^n disjuncts altogether, with n being the number of propositional variables occurring in A and $\neg A$, respectively. The union of the CDNF of A and $\neg A$ are a mapping of the truth-tables. The CDNF of A corresponds to the T-lines; each disjunct corresponds to the assignments of truth values to the propositional variables in the left part of the truth-table. Likewise, the CDNF of $\neg A$ corresponds to the F-lines. By the CDNF, one first constructs the maximal number of maximal complex disjuncts that have to be considered to identify the conditions of truth of A and $\neg A$ (= conditions of falsehood of A). The construction of CDNFs ensures that all minimal disjuncts are contained as parts in the CDNF in any case. This is because the truth of a *wff* A can only depend on the truth or falsehood of the propositional variables occurring in A. By the CDNF of A and $\neg A$, all 2^n possible combinations of their truth and falsehood are realized. Thus, all minimal disjuncts will be contained as parts of the disjuncts in the CDNFs in any case. Through this, it is impossible, for example, for the formulae (5.14) and (5.15) to be starting points to construct prime implicants. Starting from CDNFs, the reduction to RDNFs has a unique solution, as is well known. Propositional variables that do not affect the truth of A and of $\neg A$ do not occur anymore in the prime implicants. Thus, all equivalent *wff*s are reduced to the same RDNF because their CDNF will contain all prime implicants, and the process of generating the RDNF from CDNF will identify all prime implicants.

5.4.1.2 Reduced propositional pole-groups

There is a difference between the a- and b-pole-groups of a propositional formula A, and the CDNF of A and $\neg A$. The pole-groups may contain several occurrences of the same pole or contradictory poles, while minterms do not contain identical or contradictory atomic expressions (literals). This is a consequence of the intention to expand the ab-notation to predicate logic and thus not to presume internal relations between poles when constructing pole-groups, cf. p. 150. However, one receives propositional pole-groups mapping the CDNF of A and $\neg A$ one to one by including every pole only once in a pole-group, by listing every pole-group only once, and by eliminating all pole-groups containing contradictory poles. Due to PG E / PG I and P E, $\perp$ E / PG I, these eliminations can be identified as equivalent transformations. The resulting simplified pole-groups shall be called "min-pole-groups". Min-pole-groups can be minimized by recursively applying M-R, as in the case of generating the prime implicants. Again, this amounts to identifying an equivalence transformation due to $\top$ I, P E, and PG E in the system of implications between pole-groups. The resulting pole-groups will be called "prime pole-groups". The resulting lists of a- and b-pole-groups will be called, "reduced a-pole-groups," and "reduced b-pole-groups," respectively. ab-symbols contain only prime pole-groups. They consist of the reduced a- and b-pole-groups.

To solve the equivalence problem by ab-notation, the ab-symbol of a *wff* must be constructed. In propositional logic, this can be carried out by the following steps:

equivalence problem rule (EP-R.):

1. Generate the ab-diagram of A.
2. Generate the symbolizing pole-groups of A's ab-diagram.
3. Generate the min-pole-groups.
4. If a- as well as b-pole-groups remain, generate the prime-pole-groups.

As we will see, this rule also holds for elementary logic. That is why step 2 refers to *symbolizing* pole-groups.

Eliminating pole-groups with contradictory poles suffices to decide whether A is tautologous or contradictory in propositional logic. By referring to Wittgenstein's decision rule (cf. CL, p. 53 and p. 129 of this book), the following decision rules for tautologies and contradictions can be established in propositional logic:

rule of tautologies: If no b-min-pole-group remains, A is a tautology.

rule of contradictions: If no a-min-pole-group remains, A is a contradiction.

Thus, by the non-existence of a- and b-pole-groups, tautologies and contradictions, respectively, are already identified. The ab-symbol of a tautology is the empty set of b-min-pole-groups. The ab-symbol of a contradiction is the empty set of a-min-pole-groups.

The application of EP-R. is demonstrated in the following example.

EXAMPLE:

$$R \wedge (P \wedge (P \rightarrow Q) \rightarrow Q)$$

The ab-diagram is represented in figure 5.5.

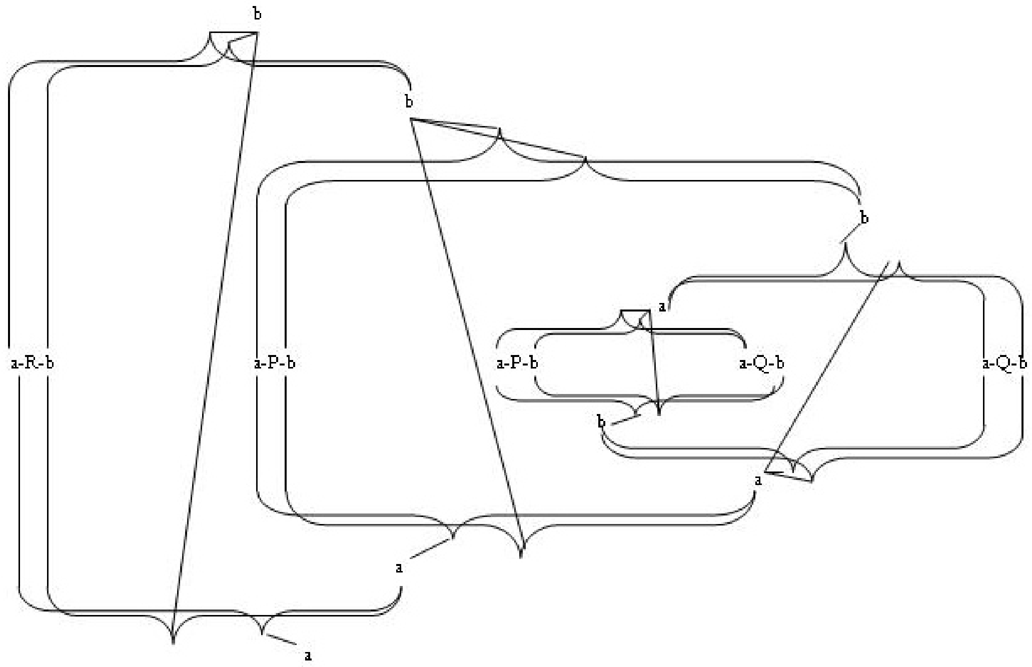

Figure 5.5: ab-diagram of $R \wedge (P \wedge (P \rightarrow Q) \rightarrow Q)$

Pole-groups:

1. a-{a-R, a-P, a-P, a-Q, a-Q},
2. b-{a-R, a-P, a-P, a-Q, b-Q},

3. a-{a-R, a-P, a-P, b-Q, a-Q},
4. a-{a-R, a-P, a-P, b-Q, b-Q},
5. a-{a-R, a-P, b-P, a-Q, a-Q},
6. b-{a-R, a-P, b-P, a-Q, b-Q},
7. a-{a-R, a-P, b-P, b-Q, a-Q},
8. b-{a-R, a-P, b-P, b-Q, b-Q},
9. a-{a-R, b-P, a-P, a-Q, a-Q},
10. a-{a-R, b-P, a-P, a-Q, b-Q},
11. a-{a-R, b-P, a-P, b-Q, a-Q},
12. a-{a-R, b-P, a-P, b-Q, b-Q},
13. a-{a-R, b-P, b-P, a-Q, a-Q},
14. a-{a-R, b-P, b-P, a-Q, b-Q},
15. a-{a-R, b-P, b-P, b-Q, a-Q},
16. a-{a-R, b-P, b-P, b-Q, b-Q},
17. b-{b-R, a-P, a-P, a-Q, a-Q},
18. b-{b-R, a-P, a-P, a-Q, b-Q},
19. b-{b-R, a-P, a-P, b-Q, a-Q},
20. b-{b-R, a-P, a-P, b-Q, b-Q},
21. b-{b-R, a-P, b-P, a-Q, a-Q},
22. b-{b-R, a-P, b-P, a-Q, b-Q},
23. b-{b-R, a-P, b-P, b-Q, a-Q},
24. b-{b-R, a-P, b-P, b-Q, b-Q},
25. b-{b-R, b-P, a-P, a-Q, a-Q},
26. b-{b-R, b-P, a-P, a-Q, b-Q},
27. b-{b-R, b-P, a-P, b-Q, a-Q},
28. b-{b-R, b-P, a-P, b-Q, b-Q},
29. b-{b-R, b-P, b-P, a-Q, a-Q},
30. b-{b-R, b-P, b-P, a-Q, b-Q},
31. b-{b-R, b-P, b-P, b-Q, a-Q},

32. b-{b-R, b-P, b-P, b-Q, b-Q}.

Min-pole-groups:

1. a-{a-R, a-P, a-Q},
4. a-{a-R, a-P, b-Q},
13. a-{a-R, b-P, a-Q},
16. a-{a-R, b-P, b-Q},

17. b-{b-R, a-P, a-Q},
20. b-{b-R, a-P, b-Q},
29. b-{b-R, b-P, a-Q},
32. b-{b-R, b-P, b-Q}.

Generating prime pole-groups:

1. TABLE	2. TABLE	3. TABLE
a-{a-R, a-P, a-Q},	a-{a-R, a-P},	a-{a-R}.
a-{a-R, a-P, b-Q},	a-{a-R, a-Q},	
a-{a-R, b-P, a-Q},	a-{a-R, b-Q},	
a-{a-R, b-P, b-Q}.	a-{a-R, b-P}.	

1. TABLE	2. TABLE	3. TABLE
b-{b-R, a-P, a-Q},	b-{b-R, a-P},	b-{b-R}.
b-{b-R, a-P, b-Q},	b-{b-R, a-Q},	
b-{b-R, b-P, a-Q},	b-{b-R, b-Q},	
b-{b-R, b-P, b-Q}.	b-{b-R, b-P}.	

All min-pole-groups of table 1 and all pole-groups of table 2 are merged in the case of the a-pole-groups as well as in the case of the b-pole-groups. Thus, the prime pole-groups are the following:

a-{a-R},
b-{b-R}.

The ab-symbol of the *wff* $R \wedge (P \wedge (P \ \rightarrow \ Q) \rightarrow Q)$ consists of these two pole-groups. It is identical to the ab-symbol of the *wff* R. The two propositional formulae, $R \wedge (P \wedge (P \ \rightarrow \ Q) \ \rightarrow \ Q)$ and R, differ in type. However, they represent the same truth function, as their identical ab-symbol makes clear.

5.4.2 Solution of the Equivalence Problem

The main difference between the pole-groups of elementary predicate logic and those of propositional logic is the fact that internal relations between poles and pole-groups of elementary predicate logic are more complicated. In propositional logic, only contradictory relations between different poles exist, and only contradictory relations and occurrences of identical poles have to be considered in minimizing pole-groups. Meanwhile, in elementary predicate logic subaltern, contrary and subcontrary relations also have to be considered.

By generating min-pole-groups in elementary predicate logic, we not only eliminate pole-groups containing contradictory poles. In general, all pole-groups containing *contrary* poles are eliminated because they do not determine consistent conditions of truth or falsehood. We call pole-groups not determining consistent conditions of truth or falsehood “contrary pole-groups”. In elementary predicate logic, contrary pole-groups are those pole-groups containing contrary poles (cf. p. 214). Thus, the general rule to generate min-pole-groups is the following:

min-rule (MIN-R.):

1. Eliminate all contrary pole-groups.
2. List every pole of a pole-group only once.
3. List every pole-group only once.

This rule comprises the rule to generate min-pole-groups in propositional logic, because contradictory poles are contrary poles. Due to this rule, tautologies and contradictions of elementary predicate logic can be identified by the *rule of tautologies* and the *rule of contradictions*, cf. p. 209. Thus, applying MIN-R. already suffices to solve the *decision problem* in elementary predicate logic.

MIN-R. 1. can be traced back to $\bot E$ by applying the seven prefix rules and P E. By the seven prefix rules, contrary poles can be reduced to contradictory poles, and by P E, all poles except the two contradictory poles can be eliminated. Finally, the pole-group containing contradictory poles can then be eliminated by $\bot E$. As every eliminated pole-group can again be introduced via PG I, the system of implications between pole-groups shows that MIN-R. 1 is an equivalence transformation as long as some pole-group remains. In addition, MIN-R. 1. can also eliminate the last remaining pole-group, implying a pair of contradictory poles. In this case, a tautology or contradiction is identified by the non-existence of a- or b-pole-groups. MIN-R. 2 and MIN-R. 3 correspond to the equivalences $A \wedge A \dashv\vdash A$ and $A \vee A \dashv\vdash A$ in predicate logic. The corresponding rules for pole-groups are derivable by the calculus of the 13 implication rules: $\{A, A\} \dashv\vdash \{A\}$ from P E and P I, $\{A\}$,$\{A\} \dashv\vdash \{A\}$ from PG E and P I.

To generate prime pole-groups from min-pole-groups, equivalent transformations must be identified that can eliminate (i) poles of one pole-group, (ii) one of two pole-groups, and (iii) poles of two pole-groups. These strategies generalize the elimination of single poles due to (α) $\{A, A, B\} \dashv\vdash \{A, B\}$ (involving P I and P E), (β) $\{B\}, \{B\} \dashv\vdash \{B\}$ (involving PG E and PG I), and (γ) $\{A, B\}, \{\overline{A}, B\} \dashv\vdash \{B\}$ (*M-R*, involving $\top$ I, P E and PG E) in propositional logic. In elementary logic, these three cases (α) to (γ) are special cases of more general ones (i) to (iii). In the case of (i), it suffices that a pole A_1 implies another pole A_2 in order to eliminate A_2 – A_1, and A_2 need not be identical. $\{A_1, A_2, B\}$ can be reduced to $\{A_1, B\}$ given $A_1 \vdash A_2$. In the case of (ii) it suffices that the poles of one pole-group $\{B_1\}$ imply the poles of the other pole-group $\{B_2\}$ in order to eliminate $\{B_1\}$ – $\{B_1\}$, and $\{B_2\}$ need not be identical. $\{B_1\}, \{B_2\}$ can be reduced to $\{B_2\}$ given $B_1 \vdash B_2$. In the case of (iii), it suffices that A and $\overline{A}$ are subcontrary – they need not be contradictory. Furthermore, it suffices that the remaining poles of pole-group B_1 imply all remaining poles B_2 of the other pole-group in order to eliminate either A or $\overline{A}$ in B_1 – B_1, and B_2 need not to be identical. $\{A, B_1\}, \{\overline{A}, B_2\}$ can be reduced to $\{B_1\}, \{\overline{A}, B_2\}$ if $A \ \diamond\!-\!\diamond\ \overline{A}$ and $B_1 \vdash B_2$.

The rules defined for (i) to (iii), plus the possibility of identifying contrary pole-groups by identifying contrary poles, cause minimization of pole-groups to depend on internal relations between *poles* only. This is possible in elementary predicate logic, as the poles here contain only one propositional function. Thus, for example, it is impossible that only two poles together are contrary to / subcontrary to / imply another pole, or that one pole itself is contrary / subcontrary (tautologous) or contains non-symbolizing properties. This is a difference between molecular predicate logic and elementary predicate logic. Therefore, in molecular logic, one must refer, among other things, to the more general rules MR1-3 defined below, presuming a complete calculus not only for elementary pole-groups, but for pole-groups or $\bigvee\bigwedge cs$ in general. However, the possibility of reducing the process of minimizing to identify internal relations between poles simplifies the generation of ab-symbols in elementary predicate logic. This is because it is not necessary to refer to the system of implications of pole-groups. Instead, it suffices to use the decision procedure described in section 5.3.6 for identifying relations of implication between poles, and to make use of the definitions of contradictory, contrary, and subcontrary poles to identify internal relations of poles, cf. section 5.3. However, the general cases (i) to (iii) are also considered if one takes into account the corresponding equivalent transformations according to the system of implications of elementary pole-groups. According to cases (i) to (iii), the following three rules of minimization can be defined:

minimization rule 1 (MR1): Minimize a pole-group if, due to P E and P I, it is equivalent with itself reduced by one pole:

$$\text{If } \{A,B\} \dashv\vdash \{B\} \text{ then } \{A,B\} \Rightarrow \{B\}$$

minimization rule 2 (MR2): Minimize a pair of pole-groups if – due to PG E and PG I – it is equivalent with one of the two pole-groups:

$$\text{If } \{B\},\{C\} \dashv\vdash \{B\} \text{ then } \{B\},\{C\} \Rightarrow \{B\}$$

minimization rule 3 (MR3): Minimize a pair of pole-groups if, due to P E and $\top$ I, it is equivalent with itself except for one element of the pair being reduced by one pole:

$$\text{If } \{B,A\},\{C\} \dashv\vdash \{B\},\{C\} \text{ then } \{B,A\},\{C\} \Rightarrow \{B\},\{C\}$$

The parenthesis refer to those rules that are necessary (but not necessarily sufficient) for identifying the relevant equivalences. The critical applications are

P I in MR1, PG E in MR2, $\top$ I in MR3. In each of these cases, these rules identify the critical direction of implications that is only due to specific internal relations between the poles of the pole-groups.

By referring to MR1-3, the prime pole-groups can be generated from the min-pole-groups according to the following procedure that starts with the min-pole-groups:

Prime rule:

1. Starting from the table of min-pole-groups, apply MR2 to the resulting table of pole-groups. If MR3 is applicable to the result of the application of MR2, go to 2.; if not, go to 3.
2. Apply MR3. Create a new table with all the pole-groups MR3 is not applied to, in addition to the results of the application of MR3. Go back to 1.
3. Apply MR1 to the resulting table of pole-groups, and go to the beginning of 3 until MR1 is not applicable any more. Finally, apply MR2 once more to the resulting pole-groups.

The resulting pole-groups, to which MR1-3 are no longer applicable, are the prime pole-groups. In elementary predicate logic, the rules defined for (i) to (iii), p. 214 can be applied instead of MR1-3.

As in the case of the Quine-McCluskey algorithm, it must be determined to which pairs of pole-groups MR3 is applied, and the identified pole-groups have to be kept for further applications of MR3. The only rule that allows elimination of pole-groups is MR2. On the other hand, contrary to the Quine-McCluskey algorithm, all pole-groups to which MR3 is no longer applicable must be transferred to the new table. First, this convention makes the procedure more transparent because prime-pole-groups can only occur in the last table as a consequence of this convention. Second this convention is also necessary because, based on the definition of MR3, it is possible that MR3 might be applicable to pole-groups that MR3 was not applicable to before. This is because, contrary to M-R in propositional logic, MR3 does not presuppose that the pole-groups are identical except for one pole. Thus, MR3 does not presume that the pole-groups contain the same number of poles. Pole-groups to which MR3 was applicable, however, need not be considered in a new table because they can be eliminated by MR2 anyway. Thus, only the results of the application of MR3, and those pole-groups to which MR3 was not applied, need to be transferred to in a new table.

There is no need to test the application of MR3 subsequent to the application of MR1, because the application of MR1 does not determine any further application of MR3. The reasons for this are the following:

- If poles $B_1 \ldots B_m$ of a pole-group B are not implied by the poles of another pole-group C, then they are not implied by these poles as a consequence of applying MR1 to B either.
- If the poles of a pole-group C do not imply poles $B_1 \ldots B_m$ of a pole-group B, then the poles of C remaining as a consequence of applying MR1 to C do not imply poles $B_1 \ldots B_m$ either.
- If two pole-groups do not contain subcontrary poles before applying MR1, they will also not contain subcontrary poles subsequent to applying MR1.

Thus, if the conditions to apply MR3 are not satisfied before applying MR1, they will also not be satisfied subsequent to MR1. Therefore, there is no need to test the application of MR3 subsequent to the application of MR1

On the contrary, application of MR3 may exclude the possibility for applying MR1. This is the case if applying MR3 eliminates poles of a pole-group B that imply other poles of B. However, the application of MR2 to the pole-group B^* resulting from the application of MR1 and to the pole-group B^{**} resulting from the application of MR3 would eliminate B^* anyway. This is because the poles of B^* imply the poles of B^{**} in this case. That is why MR1 need not be applied before MR3. Finally, applying MR2 before MR3 and MR1 ensures that only those pole-groups are minimized that are as "weak" as possible – and in this sense as "minimal" as possible – with respect to the remaining pole-groups, cf. below p. 218.

The procedure specified by the *prime rule* is a generalization of the construction of prime pole-groups in propositional logic. In propositional logic, relations of implication exist only between two occurrences of identical poles and between identical pole-groups; subcontrary relations exist only between contradictory poles. By referring to MR1-3, on the other hand, the procedure of minimizing min-pole-groups is not limited to the specific relations between occurrences of identical or contradictory poles.

Prime pole-groups do not contain any "non-symbolizing poles", i.e., poles that are irrelevant for identifying conditions of truth or falsehood of the initial formulae. In propositional logic, prime pole-groups (minimal pole-groups) can be identified by their length. A minimal pole-group is a pole-group that does not contain a part that is a sufficient condition of the truth or falsehood of the initial formula (= condition 1). This definition only suffices as long as we need not consider any relations of implications between different poles. However, in elementary predicate logic, minimal pole-groups / prime pole-groups must additionally be defined by the condition that their poles do not imply all the poles of another pole-group with the same outer pole (= condition 2). If this condition were not satisfied, then the pole-groups would not be considered minimal because they would not be the weakest truth or falsehood conditions. Any solution of the equivalence problem also implies this reading of "minimal" conditions of the truth and falsehood. The formulae $\forall xFx \vee \exists yFy$ and $\exists yFy$, for example, are equivalent. The disjuncts $\forall xFx$ and $\exists yFy$ both satisfy condition 1. Only applying MR2 ensures that condition 2 is satisfied in the case of formula $\forall xFx \vee \exists yFy$; and only by applying MR2 one does end up with identical ab-symbols. $\forall xFx \vee Fa \wedge Fb$ and $Fa \wedge Fb$ are also equivalent. Both are represented by the ab-symbol $a - \{a_1 - a - F_1, b_1 - a - F_1\}; b - \{a_1 - b - F_1\}, b - \{b_1 - b - F_1\}$. This shows that the weaker conditions are not necessarily shorter than stronger ones in predicate logic.

Application of the *equivalence problem rule*, cf. p. 209, presupposes application of the *translation rule*, cf. p. 156, to generate ab-diagrams. It presupposes the *path rule*, cf. p. 158, the *elementary variable rule*, the *quantifier ordering rule*, and the *name rule* to generate symbolizing pole-groups, cf. p. 161 to p. 162. It also presupposes the *min-rule*, cf. p. 213 to generate min-pole-groups and the *prime rule*, cf. p. 216, to generate the prime pole groups. By applying these rules successively, the equivalence problem of elementary predicate logic is solved for the same reason as in the case of propositional logic: the pole-groups generated from the ab-diagram identify a maximal combination of poles that the truth and falsehood of the initial formula can depend on. Starting from these pole-groups, the minimization procedure identifies the maximal number of minimal symbolizing pole groups, i.e. *all* prime pole groups.

On the basis of the specification of internal relations between pole-groups, it is also possible to define a procedure that further imitates the second step of the Quine-McCluskey algorithm, namely generating minimal disjunctions of minimal disjuncts. However, as the solution of the equivalence problem does not rely on this step, we refrain from defining it here.

Finally, applying the *equivalence problem rule*, and thus, the procedure of identifying the ab-symbol of a *wff* in elementary predicate logic, shall be demonstrated with an example.

EXAMPLE:

$$\forall z\exists y\forall xFxyz \wedge \forall x\exists yFxyx \wedge \forall x\forall yFxxy \ \vee\ \exists xFxxx$$

Figure 5.6 represents the ab-diagram.

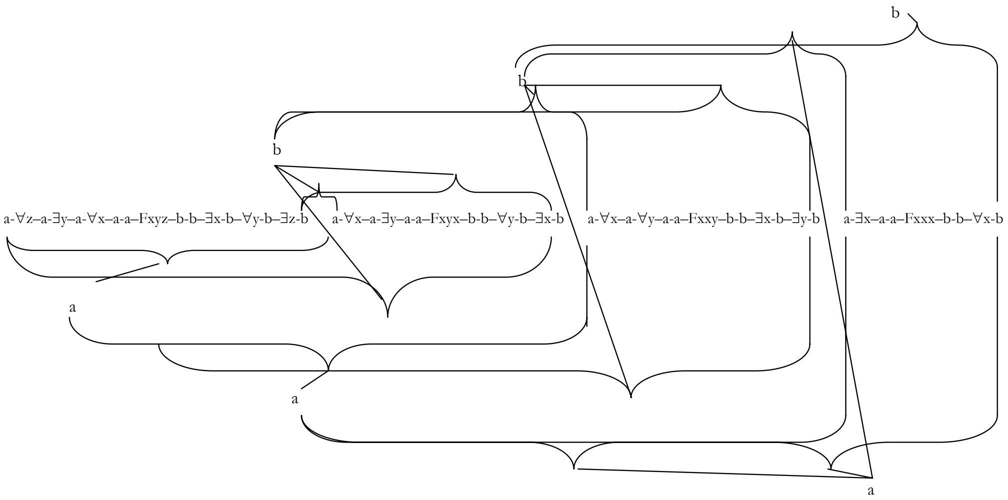

Figure 5.6: ab-diagram

Symbolizing pole-groups:

1. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall <{}^{1}_{3}\exists_2 - a - F_{123},\quad \forall <{}^{1}_{2},\forall_3 - a - F_{123},\quad \exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$
2. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall <{}^{1}_{3}\exists_2 - a - F_{123},\quad \forall <{}^{1}_{2},\forall_3 - a - F_{123},\quad \forall <^{1}_{\substack{2\\3}} - b - F_{123}\},$
3. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall <{}^{1}_{3}\exists_2 - a - F_{123},\quad \exists <{}^{1}_{2},\exists_3 - b - F_{123},\quad \exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$
4. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \exists <{}^{1}_{3}\forall_2 - b - F_{123},\quad \forall <{}^{1}_{2},\forall_3 - a - F_{123},\quad \exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$
5. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall <{}^{1}_{3}\exists_2 - a - F_{123},\quad \forall <{}^{1}_{2},\forall_3 - a - F_{123},\quad \exists <^{1}_{\substack{2\\3}} - a - F_{123}\},$
6. $b - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall <{}^{1}_{3}\exists_2 - a - F_{123},\quad \exists <{}^{1}_{2},\exists_3 - b - F_{123},\quad \forall <^{1}_{\substack{2\\3}} - b - F_{123}\},$
7. $b - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \exists <{}^{1}_{3}\forall_2 - b - F_{123},\quad \forall <{}^{1}_{2},\forall_3 - a - F_{123},\quad \forall <^{1}_{\substack{2\\3}} - b - F_{123}\},$

8. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \exists <^{1}_{3}\forall_2 - b - F_{123},\quad \exists <^{1}_{2},\exists_3 - b - F_{123},\quad \exists <^{1}_{3} - a - F_{123}\},$
9. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \forall <^{1}_{2},\forall_3 - a - F_{123},\quad \forall <^{1}_{3} - b - F_{123}\},$
10. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \exists <^{1}_{2},\exists_3 - b - F_{123},\quad \exists <^{1}_{3} - a - F_{123}\},$
11. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \exists <^{1}_{3}\forall_2 - b - F_{123},\quad \forall <^{1}_{2},\forall_3 - a - F_{123},\quad \exists <^{1}_{3} - a - F_{123}\},$
12. $b - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \exists <^{1}_{3}\forall_2 - b - F_{123},\quad \exists <^{1}_{2},\exists_3 - b - F_{123},\quad \forall <^{1}_{3} - b - F_{123}\},$
13. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \exists <^{1}_{2},\exists_3 - b - F_{123},\quad \forall <^{1}_{3} - b - F_{123}\},$
14. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \exists <^{1}_{3}\forall_2 - b - F_{123},\quad \forall <^{1}_{2},\forall_3 - a - F_{123},\quad \forall <^{1}_{3} - b - F_{123}\},$
15. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \exists <^{1}_{3}\forall_2 - b - F_{123},\quad \exists <^{1}_{2},\exists_3 - b - F_{123},\quad \exists <^{1}_{3} - a - F_{123}\},$
16. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \exists <^{1}_{3}\forall_2 - b - F_{123},\quad \exists <^{1}_{2},\exists_3 - b - F_{123},\quad \forall <^{1}_{3} - b - F_{123}\}.$

Figure 5.7, p. 222, represents the relations of implication between the symbolizing poles with the innermost a-pole. Figure 5.8, p. 223, represents the relations of implication between the symbolizing poles with the innermost b-pole.

Due to the definition of contradictory and contrary poles, the contrary poles can be identified, cf. table 5.15, p. 223. The first four pairs of poles of the table are contradictory, and thus not only contrary, but also subcontrary.

Thus, lines 2,4,7-9,11,12, and 14 contain contrary poles. According to the *min-rule*, the pole-groups of these lines can be eliminated. Therefore, the following min-pole-groups remain:

1. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \forall <^{1}_{2},\forall_3 - a - F_{123},\quad \exists <^{1}_{3} - a - F_{123}\},$
3. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \exists <^{1}_{2},\exists_3 - b - F_{123},\quad \exists <^{1}_{3} - a - F_{123}\},$
5. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \forall <^{1}_{2},\forall_3 - a - F_{123},\quad \exists <^{1}_{3} - a - F_{123}\},$
10. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \exists <^{1}_{2},\exists_3 - b - F_{123},\quad \exists <^{1}_{3} - a - F_{123}\},$
15. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \exists <^{1}_{3}\forall_2 - b - F_{123},\quad \exists <^{1}_{2},\exists_3 - b - F_{123},\quad \exists <^{1}_{3} - a - F_{123}\},$

6. $b - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \exists <^{1}_{2},\exists_3 - b - F_{123},\quad \forall <^{1}_{3} - b - F_{123}\},$
13. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall <^{1}_{3}\exists_2 - a - F_{123},\quad \exists <^{1}_{2},\exists_3 - b - F_{123},\quad \forall <^{1}_{3} - b - F_{123}\},$

16. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \exists < {}^{1}_{3}\forall_2 - b - F_{123},\quad \exists < {}^{1}_{2},\exists_3 - b - F_{123},\quad \forall <^{1}_{3} - b - F_{123}\}.$

The *prime rule* presupposes the identification of subcontrary poles, cf. table 5.16, p. 224. This is due to the equivalence of MR3 and the rule that $\{A, B_1\}, \{\overline{A}, B_2\}$ can be reduced to $\{B_1\}, \{\overline{A}, B_2\}$ in elementary predicate logic if $A \ \diamond - \diamond \ \overline{A}$ and $B_1 \vdash B_2$, cf. p. 214. According to their definition, the subcontrary poles can be identified by contradictory and subaltern poles.

In the following, the generation of prime pole-groups is represented by tables, beginning with the table of min-pole-groups. MR2 or the equivalent rule that $\{B_1\}, \{B_2\}$ can be reduced to $\{B_2\}$ in elementary predicate logic given $B_1 \vdash B_2$, cf. p. 214, do not change the min-pole-groups in this case. In the second column of the tables, the rules and their application to certain pole-groups are displayed, which leads to the pole-groups of the new table. Only the resulting tables of a certain step are listed. Thus, applications of MR2 are not specially indicated. Instead, only the pole-groups of a table remaining as consequence of applying MR2 are mentioned. The application of MR1 is based on its equivalent rule that $\{A_1, A_2, B\}$ can be reduced to $\{A_1, B\}$ in elementary predicate logic given $A_1 \vdash A_2$, p. 214. This, again, presupposes the identification of the relations of implication between the poles, cf. figure 5.7, p. 222 and figure 5.8, p. 223.

a-pole-groups:

TABLE 1

1. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\ \forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \forall < {}^{1}_{2},\forall_3 - a - F_{123}, \exists <^{1}_{3} - a - F_{123}\},$
2. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\ \forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists < {}^{1}_{2},\exists_3 - b - F_{123}, \exists <^{1}_{3} - a - F_{123}\},$
3. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \forall < {}^{1}_{2},\forall_3 - a - F_{123}, \exists <^{1}_{3} - a - F_{123}\},$
4. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists < {}^{1}_{2},\exists_3 - b - F_{123}, \exists <^{1}_{3} - a - F_{123}\},$
5. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \exists < {}^{1}_{3}\forall_2 - b - F_{123},\ \exists < {}^{1}_{2},\exists_3 - b - F_{123}, \exists <^{1}_{3} - a - F_{123}\}.$

TABLE 2

6. MR3: 1/2 $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\ \forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists <^{1}_{3} - a - F_{123}\},$
7. MR3.: 1/3 $a - \{\forall < {}^{1}_{3}\exists_2 - a - F_{123}, \forall < {}^{1}_{2},\forall_3 - a - F_{123}, \exists <^{1}_{3} - a - F_{123}\},$
8. MR3: 2/4 $a - \{\forall < {}^{1}_{3}\exists_2 - a - F_{123}, \exists < {}^{1}_{2},\exists_3 - b - F_{123}, \exists <^{1}_{3} - a - F_{123}\},$
9. MR3: 3/4 $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists <^{1}_{3} - a - F_{123}\},$
10. MR3: 4/5 $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \exists < {}^{1}_{2},\exists_3 - b - F_{123}, \exists <^{1}_{3} - a - F_{123}\}.$

TABLE 3

11. MR3: 6/9 $a - \{\forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists <^{1}_{3} - a - F_{123}\},$
12. MR3 8/10 $a - \{\exists < {}^{1}_{2},\exists_3 - b - F_{123}, \exists <^{1}_{3} - a - F_{123}\}.$

TABLE 4

13. MR3: 11/12 $a - \{\exists <^{1}_{3} - a - F_{123}\}.$

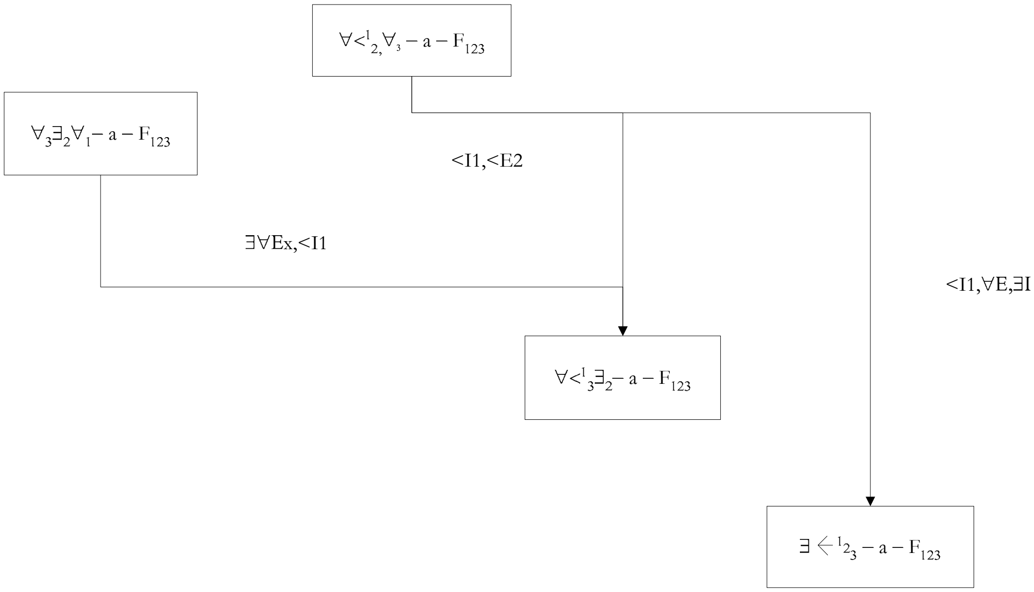

Figure 5.7: Relations of implication between symbolizing poles with the innermost a-pole

b-pole-groups:

TABLE 1

1. $b - \{\forall_3\exists_2\forall_1 - a - F_{123},\ \ \forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists < {}^{1}_{2},\exists_3 - b - F_{123}, \forall <\!{}^{1}_{3} - b - F_{123}\},$
2. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists < {}^{1}_{2},\exists_3 - b - F_{123}, \forall <\!{}^{1}_{3} - b - F_{123}\},$
3. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \exists < {}^{1}_{3}\forall_2 - b - F_{123},\ \exists < {}^{1}_{2},\exists_3 - b - F_{123}, \forall <\!{}^{1}_{3} - b - F_{123}\}.$

TABLE 2

4. MR3: 1/2 $b - \{\forall < {}^{1}_{3}\exists_2 - a - F_{123},\ \exists < {}^{1}_{2},\exists_3 - b - F_{123}, \forall <\!{}^{1}_{3} - b - F_{123}\},$
5. MR3: 2/3 $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \exists < {}^{1}_{2},\exists_3 - b - F_{123}, \forall <\!{}^{1}_{3} - b - F_{123}\}.$

TABLE 3

6. MR3.: 4/5 $b - \{\exists < {}^{1}_{2},\exists_3 - b - F_{123}, \forall <\!{}^{1}_{3} - b - F_{123}\}.$

TABLE 4

7. MR1: 6 $b - \{\forall <\!{}^{1}_{3} - b - F_{123}\}.$

Thus, the prime pole-groups are $a - \{\exists <\!{}^{1}_{3} - a - F_{123}\}$ and $b - \{\forall <\!{}^{1}_{3} - b - F_{123}\}$. Therefore, the ab-symbol of the initial formula is the following:

$$a - \{\exists <\!{}^{1}_{3} - a - F_{123}\},$$
$$b - \{\forall <\!{}^{1}_{3} - b - F_{123}\}.$$

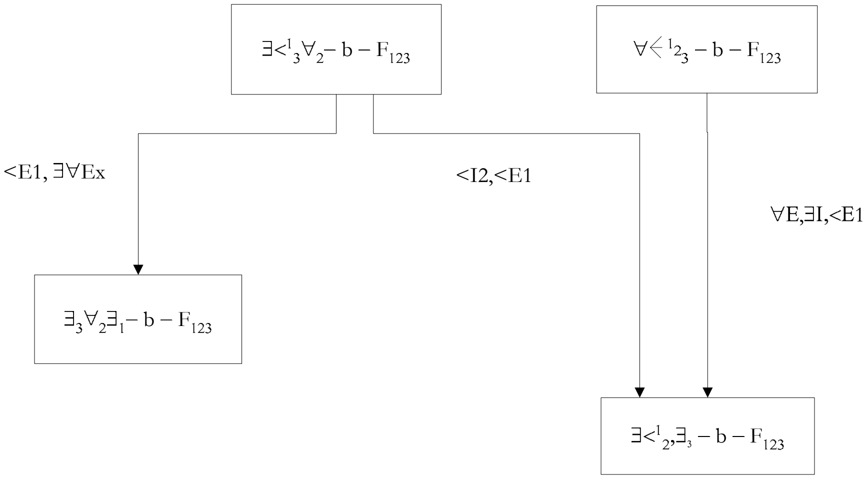

Figure 5.8: Relations of implication between symbolizing poles with the innermost b-pole

$$
\begin{array}{lcl}
\forall_3 \exists_2 \forall_1 - a - F_{123} & \lhd\!-\!\rhd & \exists_3 \forall_2 \exists_1 - b - F_{123} \\
\forall <\, ^1_3 \exists_2 - a - F_{123} & \lhd\!-\!\rhd & \exists <\, ^1_3 \forall_2 - b - F_{123} \\
\forall <\, ^1_2, \forall_3 - a - F_{123} & \lhd\!-\!\rhd & \exists <\, ^1_2, \exists_3 - b - F_{123} \\
\exists \lessdot^1_3 - a - F_{123} & \lhd\!-\!\rhd & \forall \lessdot^1_3 - b - F_{123} \\
\forall_3 \exists_2 \forall_1 - a - F_{123} & \lhd\!-\!\rhd & \exists <\, ^1_3 \forall_2 - b - F_{123} \\
\forall <\, ^1_2, \forall_3 - a - F_{123} & \lhd\!-\!\rhd & \exists <\, ^1_3 \forall_2 - b - F_{123} \\
\forall <\, ^1_2, \forall_3 - a - F_{123} & \lhd\!-\!\rhd & \forall \lessdot^1_3 - b - F_{123}.
\end{array}
$$

Table 5.15: Contrary pairs of poles

$\forall_3\exists_2\forall_1 - a - F_{123}$	$\diamond\!-\!\diamond$	$\exists_3\forall_2\exists_1 - b - F_{123}$
$\forall < {}^{1}_{3}\exists_2 - a - F_{123}$	$\diamond\!-\!\diamond$	$\exists < {}^{1}_{3}\forall_2 - b - F_{123}$
$\forall < {}^{1}_{2},\forall_3 - a - F_{123}$	$\diamond\!-\!\diamond$	$\exists < {}^{1}_{2},\exists_3 - b - F_{123}$
$\exists <^{1}_{\substack{2\\3}} - a - F_{123}$	$\diamond\!-\!\diamond$	$\forall <^{1}_{\substack{2\\3}} - b - F_{123}$
$\forall < {}^{1}_{3}\exists_2 - a - F_{123}$	$\diamond\!-\!\diamond$	$\exists_3\forall_2\exists_1 - b - F_{123}$
$\forall < {}^{1}_{3}\exists_2 - a - F_{123}$	$\diamond\!-\!\diamond$	$\exists < {}^{1}_{2},\exists_3 - b - F_{123}$
$\exists <^{1}_{\substack{2\\3}} - a - F_{123}$	$\diamond\!-\!\diamond$	$\exists < {}^{1}_{2},\exists_3 - b - F_{123}$

Table 5.16: Subcontrary pairs of poles

Chapter 6

Molecular Predicate Logic

Molecular predicate logic comprises the entire realm of pure predicate logic. Thus, unlike elementary predicate logic, it includes dyadic sentential connectives within the scope of quantifiers. This chapter elaborates on expanding the ab-notation to the whole realm of molecular predicate logic. First, we define the ab-syntax in general and its relation to arbitrary predicate formulae (section 6.1). Then we go on to specify a procedure to convert predicate formulae to pole-groups (section 6.2). We will accomplish this based on an equivalence transformation of *wff*s to disjunctions of conjunctions of closed structures $\bigvee \bigwedge cs$ that correspond to pole-groups. Eventually, we show how to explain the conditions of truth and falsehood of predicate formulae on this basis (section 6.3). Thus, we solve the problem of semantics. In section 6.4 we then define a calculus to identify all minimal valid syntactical variations between pole-groups, or rather $\bigvee \bigwedge cs$ (section 6.4). We show how to do this by generalizing the 13 rules of implication defined for elementary predicate logic. Thus, we solve the problem of implication. On this basis, we finally solve the problem of equivalence in section **??** along the same lines as in elementary predicate logic.

6.1 ab-syntax

In this section, we first show how to expand the syntax of the ab-notation from elementary predicate logic to the entire realm of predicate logic using plain examples. We then go on to specify these considerations, originally based upon examples, by providing both general and precise definitions. However, we will start by introducing the concept of closed structures as the unanalysable expressions of predicate logic.

6.1.1 Closed Structures

As explained on p. 145, the main idea of developing the programme of New Logic in the realm of molecular logic is to minimize the scope of quantifiers. Instead of prioritizing prenex normal forms, priority is placed on disjunctive normal forms. Thus, PN-laws (cf. appendix, p. 337), compared to the generation of prenex normal forms, are applied in the opposite direction . For the sake of simplicity, we assume in the following that only $\wedge$ and $\vee$ are used as dyadic sentential connectives in *wff*s, and that negation signs only occur directly to the left of propositional functions. We abbreviate formulae satisfying these two conditions with "NNF" (negation normal forms). By A, DM$\wedge$, DM$\vee$, DN, and *definitions of quantifiers*, any formula can be converted to an NNF. From this point forward, only *wff*s in terms of NNF are considered. Furthermore, in section 6.2, we demonstrate that any NNF can be converted to a disjunction of conjunctions of closed structures ($\bigvee\bigwedge cs$). Predicate logic can be completely mapped onto this subclass. The process of generating $\bigvee\bigwedge cs$ aims to realize the idea of applying PN-laws for the sake of minimizing the scope of quantifiers to a maximal extent. We call formulae in which the scope of any quantifier cannot be minimized any further by applying PN-laws "closed structures." They are defined as follows:

closed structures:

1. Any NNF not containing any conjunction or disjunction is a closed structure.
2. NNF containing conjunctions or disjunctions are closed structures iff they satisfy the following conditions:
   (a) Any conjunction with n conjuncts ($n > 1$) is preceded by a sequence of existential quantifiers with minimal length 1, and all n conjuncts contain each variable of the existential quantifiers of that sequence.
   (b) Any disjunction with n disjuncts ($n > 1$) is preceded by a sequence of universal quantifiers with minimal length 1, and all n disjuncts contain each variable of the universal quantifiers of that sequence.
3. Closed structures are only *wff* that satisfy either 1. or 2.

Closed structures, according to 1, are the closed structures of elementary predicate logic (ecs). The following table contains examples of closed structures.

P
$\exists x \forall y \neg Fxy$
$\forall x \exists y (Fxy \wedge \neg Gxy)$
$\exists y (\exists x Fxy \wedge Gy)$
$\exists x \forall y \exists z (Fyz \wedge \neg Fzx)$
$\exists y (\exists x (Fx \wedge Gxy) \wedge Hy)$
$\exists x (\forall y Fxy \wedge \exists z Gxz \wedge Hx)$
$\exists y \forall x (Fxy \vee Gxy)$
$\forall x (\exists y Fxy \vee \exists y Gxy)$
$\forall y (Fy \vee \exists x (Gxy \wedge Hx))$
$\exists y (Fy \wedge \forall x (Gxy \vee Hx))$
$\forall x \exists y (Fxy \wedge \forall z (Gzy \vee Hxz))$
$\exists y (\forall x (Fxy \vee Gx) \wedge \exists z Hzy)$
$\exists y_1 (\forall x_2 (\exists z (Gy_1 z \wedge Hzx_2) \vee \forall x_1 Ix_2 x_1) \wedge Fy_1)$
$\exists y_1 \forall x (Fxy_1 \vee Fy_1 x \vee \exists y_2 (\forall z Gzxy_2 \wedge \neg Gy_2 y_2 y_2))$
Table 6.1: closed structures

The following table contains only formulae that are not closed structures.

$P \wedge Q$
$\forall x Fx \wedge \forall y \neg Fy$
$\forall x \forall y (Fx \vee Gy)$
$\exists x \exists y (Fxy \wedge Gy)$
$\forall x (Fx \wedge \neg Fx)$
$\exists x (Fx \vee Gx)$
$\exists x \forall y (Fxy \wedge Gxy)$
$\exists y (\exists x ((Fx \wedge Hy) \wedge Gxy))$
$\exists z \exists y \forall x ((Fxy \vee Gx) \wedge Hzy)$
Table 6.2: No closed structures

$$\exists y \exists z(\forall x(Fxy \vee Gx) \wedge Hzy)$$
$$\forall x \exists y(Fxy \vee Gxy)$$
$$\forall x \forall y((Fx \wedge Gy) \vee (Hx \wedge Iy))$$
$$\forall x \exists y(Fxy \vee (Gxy \wedge Hxy))$$
$$\exists x \exists y \forall z((Fxy \wedge Gz) \vee Hz)$$
$$\exists x_1 \forall x_2 \exists x_3 \forall x_4((Fx_1x_2 \wedge \neg Gx_1x_3x_4 \wedge Hx_3) \vee (Ix_4 \wedge \neg Ix_4))$$
$$\forall x_3 \forall x_1(\exists y_1 Fx_1y_1 \vee \exists y_2(Fx_1y_2 \wedge \neg Fy_2y_2) \vee Fx_1x_3)$$

Table 6.2: No closed structures

Closed structures are also definable inductively. The inductive definition is anchored on the common notion of atomic propositions: Any propositional variable and any formula composed of a predicate letter followed by names is an atomic proposition. By $\exists \mu A(t/\mu)$ we refer to the formula that is generated from $A(t)$ by replacing all occurrences of the name t with the variable μ which is bound by the existential quantifier. The same holds likewise for $\forall \mu A(t/\mu)$.

Inductive definition of closed structures:

1. Any atomic proposition and any negated atomic proposition are closed structures.
2. If $A(t)$ is a closed structure, then $\exists \mu A(t/\mu)$ and $\forall \mu A(t/\mu)$ are closed structures.
3. If $A(t)$, $B(t)$ are closed structures, then $\exists \mu(A(t/\mu) \wedge B(t/\mu))$ is a closed structure.
4. If $A(t)$, $B(t)$ are closed structures, then $\forall \mu(A(t/\mu) \vee B(t/\mu))$ is a closed structure.
5. *wff* s are only closed structures according to 1 to 4.

Closed structures correspond to the complex poles of the pole-group notation. Any *wff* can be converted to a $\bigvee \bigwedge cs$ (cf. section 6.2). Thus, *wff* s are conceivable as truth-functions of closed structures. As described on p. 110, we base the possibility of constructing the totality of truth functions in predicate logic upon this fact. Closed structures replace the concept of atomic propositions in New

Logic. Instead of referring to the totality of atomic propositions, one has to base the "general form of a truth function" on the totality of closed structures: $\overline{cs}$ has to replace $\overline{p}$ in Wittgenstein's definition in TLP 6.

$\bigvee \bigwedge cs$ are the pendent to pole-groups; the multiplicity of the former's syntax corresponds to the multiplicity of the latter's syntax. This motivates using the syntactical features of the ab-notation in molecular predicate logic, which is demonstrated in the following subsection.

6.1.2 Expanding the syntax

To define the syntax of ab-notation for the entire realm of predicate logic, we must identify the syntactical features that are necessary to express the truth conditions of arbitrary predicate formulae. This will first be demonstrated with examples in this subsection. The general features of the ab-syntax will be specified and explained in future sections.

By applying PN-laws, the formula

$$\exists x(Fx \vee Gx) \tag{6.1}$$

can be reduced to the equivalent formula $\bigvee \bigwedge cs$

$$\exists x Fx \vee \exists x Gx \tag{6.2}$$

of elementary predicate logic. Thus, both formulae can be represented by the same pole-groups. Therefore, no syntactical features other than those of elementary predicate logic are needed to express the truth conditions of formula (6.1). Simply, the rules for converting predicate formulae to pole-groups must be expanded to represent the equivalent formulae (6.1) and (6.2) by the same pole-groups. By doing this, formulae of molecular predicate logic equivalent to formulae of elementary predicate logic are reduced to formulae of elementary predicate logic.

On the contrary, the scope of the existential quantifier in formula

$$\exists x(Fx \wedge Gx) \tag{6.3}$$

cannot be minimized any further by equivalence rules. Thus, (6.3) is not reducible to an equivalent formula of elementary predicate logic. (6.3) is a closed structure of molecular predicate logic. As was shown in the simple formula $\forall x Fx$,Wittgenstein bases the translation of predicate formula to expressions of the ab-notation on the explanation of their truth conditions (cf. p. 139 and NL, p. 95f). The rules of the ab-notation of elementary predicate logic arise from generalizing and expanding the symbolizing properties needed to properly express the truth conditions of simple formulae. We proceed analogously for the case of closed structures of molecular predicate logic. Referring to the paraphrases of the truth conditions of elementary predicate formulae, the conditions of truth of (6.3) are to be paraphrased as follows:

- $\exists x(Fx \wedge Gx)$ is true iff some object, the same at the 1. position of the 1-tuples of $\Im(F)$ and the 1. position of the 1-tuples of $\Im(G)$, satisfies $\Im(F)$ and $\Im(G)$.

To represent these truth conditions in the ab-notation of elementary predicate logic, the numbers of argument positions of *different* propositional functions must be connected by closed forks as well. Thus, the pole

$$\exists < \begin{smallmatrix} 1 & -a-F_1 \\ 1 & -a-G_1 \end{smallmatrix} \tag{6.4}$$

represents the truth conditions of the formula $\exists x(Fx \wedge Gx)$. Therefore, to represent the truth conditions of $\exists x(Fx \wedge Gx)$, it is not necessary to introduce a new kind of symbol. It suffices to expand the use of closed forks such that they connect numbers of argument positions of different propositional functions.

In contrast, a new symbol must be introduced when the conditions of falsehood of the closed structure (6.3) are considered as well. One cannot simply exchange the quantifiers and innermost poles to express the conditions of falsehood, as in case of elementary poles. $\forall < \begin{smallmatrix} 1 & -b-F_1 \\ 1 & -b-G_1 \end{smallmatrix}$ would not represent the conditions of falsehood of (6.3) properly. Identifying the conditions of falsehood by $\forall < \begin{smallmatrix} 1 & -b-F_1 \\ 1 & -b-G_1 \end{smallmatrix}$ would mean that $\exists x(Fx \wedge Gx)$ is false iff all objects, *the same* at the 1. position of the 1-tuples of $\Im(F)$ and the 1. position of the 1-tuples of $\Im(G)$, do not satisfy $\Im(F)$ and $\Im(G)$. Yet, this is an inadequate paraphrase of the conditions of falsehood of (6.3), because (6.3) is, for example, also false for $I = \{c_1, c_2\}, \Im(F) = \{c_1\}, \Im(G) = \{c_2\}$. In this case, c_2 does not satisfy $\Im(F)$, whereas c_1 does not satisfy $\Im(G)$. Thus, the objects not satisfying $\Im(F)$ and $\Im(G)$ need not be the same, and it is not necessary that all objects do not satisfy $\Im(F)$ as well as $\Im(G)$. This also becomes clear when taking into account that the negation of (6.3)

$$\neg\exists x(Fx \wedge Gx) \tag{6.5}$$

is equivalent to the closed structure

$$\forall x(\neg Fx \vee \neg Gx). \tag{6.6}$$

In the case of (6.6), the sentential connective is exchanged compared to (6.5).

As in formula (6.3), the scope of the quantifier in formula (6.6) cannot be minimized any further by applying equivalence rules. (6.6) cannot be converted to an equivalent formula of elementary predicate logic. The proper paraphrase of the conditions of falsehood of (6.3) and (6.6) according to the ab-notation, is the following:[1]

- $\exists x(Fx \wedge Gx)$ is false / $\forall x(\neg Fx \vee \neg Gx)$ is true iff all objects, *distributed among* the 1. position of the 1-tuples of $\Im(F)$, and the 1. position of the 1-tuples of $\Im(G)$, do not satisfy $\Im(F)$ and $\Im(G)$.

The way the objects are distributed among the argument positions of propositional functions is arbitrary. For example, $\exists x(Fx \wedge Gx)$ is false, and equivalently, $\forall x(\neg Fx \vee \neg Gx)$ is true if $I = \{c_1,c_2\}$, $\Im(F) = \{\}$. In this case, the values of $\Im(G)$ are arbitrary. This is because all objects are distributed among $\Im(F)$ and $\Im(G)$ such that they do not satisfy $\Im(F)$ and $\Im(G)$, simply due to the fact that all objects do not satisfy $\Im(F)$.

To express the fact that all objects of the domain are *distributed among* certain positions of tuples of different propositional functions, the syntax of the ab-notation of elementary predicate logic must be expanded. In addition to closed forks, open forks must be introduced, connecting the numbers of argument places succeeding universal quantifiers. Thus, the conditions of falsehood of $\exists x(Fx \wedge Gx)$, and equivalently, the conditions of truth of $\forall x(\neg Fx \vee \neg Gx)$, can be expressed by the following pole:

$$\forall <^{1\ -b-F_1}_{1\ -b-G_1} \tag{6.7}$$

Thus, to represent truth conditions of closed structures with universal quantifiers preceding disjunctions, open forks need be introduced.

In closed structures, sequences of universal quantifiers not preceding a negated (or non-negated) propositional function do not only occur above disjunctions but

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[1] On p. 272 we will explain why we do not choose a paraphrase based on the syntax of *wff*s. For example, "$\forall x(\neg Fx \vee \neg Gx)$ is true iff, for all objects: not Fx or not Gx".

may also occur above existential quantifiers. In this case, open forks succeeding universal quantifiers are also needed to symbolize the truth conditions of closed structures, if the bound variables of the universal quantifiers occur in different disjuncts. This is demonstrated in the following example. The truth conditions of the closed structure $\forall x \exists y \forall z(Fxyz \vee Gxyz)$ are expressed adequately by the following pole:

$$\forall <^{1}_{1} \exists <{}^{2}_{2} \forall <^{3\ -a-F_{123}}_{3\ -a-G_{123}}. \tag{6.8}$$

The following $\Im$ is a model of the formula:

$$\begin{array}{lcl} I & = & \{c_1,c_2\}, \\ \Im(F) & = & \{(c_1,c_1,c_1),(c_2,c_1,c_2),(c_2,c_2,c_2)\}, \\ \Im(G) & = & \{(c_1,c_2,c_1)\}. \end{array}$$

This model satisfies the structural properties identified by $\forall <^{1}_{1} \exists <{}^{2}_{2} \forall <^{3\ -a-F_{123}}_{3\ -a-G_{123}}$:

All objects, distributed among the 1. position of the 3-tuples of $\Im(F)$ and the 1. position of the 3-tuples of $\Im(G)$, combined with some object, the same at the 2. position of the 3-tuples of $\Im(F)$ and the 2. position of the 3-tuples of $\Im(G)$, combined with all objects, distributed among the 3. position of the 3-tuples of $\Im(F)$ and the 3. position of the 3-tuples of $\Im(G)$, satisfy $\Im(F)$ and $\Im(G)$.

It is not necessary for all objects to occur at the 1. position or to occur at the 3. position of $\Im(G)$, as the mentioned model demonstrates. That is why open forks have to be used in both universal quantifiers.

Open forks are not required to connect numbers of argument positions occurring in the *same* propositional function, nor are they required after *existential quantifiers*. This, again, becomes clear through consideration of the conditions of truth and falsehood of closed structures. Unless the variable bound by a universal quantifier does not occur in different disjuncts of closed structures, all objects must satisfy the argument positions occupied by the bound variable. Therefore, open forks of universal quantifiers do not connect numbers of argument positions of the same propositional function. If an existential quantifier[2] directly precedes a disjunction, it is always possible to minimize its scope by applying PN-laws.

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[2] More precisely, we must say: “If an existential quantifier $\exists\mu$ is part of a sequence of existential quantifiers that directly precedes a disjunction, it is always possible to minimize the scope of $\exists\mu$ by applying PN-laws.” For simplicity, we refrain, here and in the following, from referring to sequences of quantifiers.

If an existential quantifier precedes a conjunction or a negated (or non-negated) propositional function in a closed structure, and if the variable bound by the existential quantifier occupies several argument positions in the conjunction or propositional function, then this means that the *same* object must satisfy these positions. Furthermore, if an existential quantifier $\exists \mu$ precedes a universal quantifier $\forall \nu$ that precedes a disjunction of a closed structure the formula is also only true if the *same* object satisfies those argument positions that the variable μ occupies. This is true even if μ occurs in both disjuncts. The formula $\exists x \forall y (Fxy \lor Gxy)$, for example, is only true if the *same* object at the 1. position of the 2-tuples of $\Im(F)$ and at the 1. position of the 2-tuples of $\Im(G)$ combined with all objects, distributed among the 2. position of the 2-tuples of $\Im(F)$ and the 2. position of the 2-tuples of $\Im(G)$, satisfies $\Im(F)$ and $\Im(G)$. That means, for example, $I = \{c_1, c_2\}, \Im(F) = \{(c_1,c_1)\}, \Im(G) = \{(c_2,c_2)\}$ is not a model, whereas $I = \{c_1, c_2\}, \Im(F) = \{(c_1,c_1)\}, \Im(G) = \{(c_1,c_2)\}$ is a model of the formula, because only in the latter $\Im$ does the same object occur at the 1. place of both tuples. Thus, in relation to existential quantifiers, only one syntactical feature is needed expressing that the *same* object must occur at certain places of tuples of $\Im(\varphi)$, such that an $\Im$ is a model or a counter-model of the initial formula. On the contrary, in relation to universal quantifiers, two different syntactical features are needed to express both that the *same* objects occur at certain positions of tuples of $\Im(\varphi)$, and that all objects of the domain are *distributed among* certain positions of tuples of $\Im(\varphi)$, such that an $\Im$ is a model or counter-model of the initial formula. This difference is accounted for by using closed, as well as open, forks subsequent to universal quantifiers.

Closed forks succeeding universal quantifiers that connect numbers of argument positions of *different* propositional functions are only required if the variable bound by an universal quantifier occurs in different conjuncts of a conjunction, and the universal quantifier precedes an existential quantifier in a closed structure. Otherwise, the variable bound by the universal quantifier either occurs in different disjuncts, such that open forks must be used, or the scope of the universal quantifier could be minimized by PN-laws. The formula $\forall x (Fx \land Gx)$, for example, is equivalent to the formula $\forall x Fx \land \forall x Gx$ of elementary predicate logic. In contrast, the two following formulae are not equivalent:

$$\forall x \exists y (Fxy \land Gxy) \tag{6.9}$$

$$\forall x_1 \forall x_2 \exists y (Fx_1 y \land Gx_2 y). \tag{6.10}$$

The truth conditions of (6.9) must be explained by pole (6.11), whereas the truth conditions of (6.10) must be explained by pole (6.12):

$$\forall < {}^{1}_{1}\exists < \begin{array}{l} 2\ {-a-}F_{12} \\ 2\ {-a-}G_{12} \end{array}, \tag{6.11}$$

$$\begin{array}{l} \forall_1 \\ \quad\forall_1 \end{array}\exists < \begin{array}{l} 2\ {-a-}F_{12} \\ 2\ {-a-}G_{12} \end{array}. \tag{6.12}$$

That means, (6.9) is true iff all objects, the same at the 1. position of the 2-tuples of $\Im(F)$ and the 1. position of the 2-tuples of $\Im(G)$, combined with some object, the same at the 2. position of the 2-tuples of $\Im(F)$ and the 2. position of the 2-tuples of $\Im(G)$, satisfy $\Im(F)$ and $\Im(G)$. On the contrary, (6.10) is true iff all objects at the 1. position of the 2-tuples of $\Im(F)$ combined with all objects at the 1. position of the 2-tuples of $\Im(G)$, combined with some object, the same at the 2. position of the 2-tuples of $\Im(F)$ and the 2. position of the 2-tuples of $\Im(G)$, satisfy $\Im(F)$ and $\Im(G)$. That means the following $\Im$ is a model of (6.9), but a counter-model of (6.10):

$$\begin{array}{lcl} I & = & \{c_1,c_2\}, \\ \Im(F) & = & \{(c_1,c_1),(c_2,c_2)\}, \\ \Im(G) & = & \{(c_1,c_1),(c_2,c_2)\}. \end{array}$$

In contrast, the following $\Im$ is also a model of (6.10):

$$\begin{array}{lcl} I & = & \{c_1,c_2\}, \\ \Im(F) & = & \{(c_1,c_1),(c_2,c_2)\}, \\ \Im(G) & = & \{(c_1,c_1),(c_2,c_1),(c_1,c_2),(c_2,c_2)\}. \end{array}$$

This difference between the truth conditions of (6.9) and (6.10) is identified by the difference in the respective poles concerning the use of a closed fork.

Using closed forks connecting numbers of argument positions of different propositional functions, and introducing open forks succeeding universal quantifiers, are the only syntactical additions to elementary predicate logic needed to express the conditions of truth and falsehood of any predicate formula within the ab-notation. This follows from the facts that any *wff* is convertible to an equivalent $\bigvee\bigwedge cs$ (section 6.2) and that $\bigvee\bigwedge cs$ are translatable into pole-groups (section 6.1.3) composed of poles with the mentioned syntactical features. By referring to the semantics of predicate logic, one can argue that the syntactical features of the ab-notation are necessary and sufficient to express the conditions of truth and falsehood of *wff*s by considering that *wff*s specify structures of $\Im$ that are identified by the syntactical features of the ab-notation (cf. section 6.3).

These general considerations shall be made precise in the following sections, which will cover the following topics:

1. Section 6.1.3 defines pole-groups syntactically and specifies how they can be translated to $\bigvee\bigwedge cs$ and v.v.
2. Section 6.2 defines a procedure to convert *wff*s to $\bigvee\bigwedge cs$ by minimizing the scope of quantifiers as much as possible by applying PN-laws.
3. Section 6.3 defines the general rules of paraphrasing pole-groups. On this basis, we will define to what extent pole-groups identify structures of $\Im$, and also how it is possible to construct systematically models and counter-models of *wff*s on the basis of their pole-groups.

6.1.3 Definitions

We call the pole-groups of the pole-group notation "ab-expressions". The ab-symbols are a special kind of "ab-expressions", namely those which solve the equivalence problem. ab-expressions and $\bigvee\bigwedge cs$ are translatable into each other.

In this section we define syntactically

- ab-expressions (6.1.3.1),
- how to translate ab-expressions into $\bigvee\bigwedge cs$ (6.1.3.2),
- how to translate $\bigvee\bigwedge cs$ into ab-expressions (6.1.3.3).

6.1.3.1 ab-expressions

The alphabet of ab-expressions consists of the following signs:

propositional variables:

1. 'P', 'Q', 'R', 'S', 'T', 'U'.
2. 'P_1'.
3. If 'P_i' is a propositional variable, then 'P_{i+1}' is a propositional variable.
4. Propositional variables are only those signs generated according to 1.-3.

quantifiers: '$\exists$', '$\forall$'.

forks: '$<$', ' $<$'.

names:

1. 'a', 'b', 'c', 'd'.
2. 'a''.
3. If 'a^k' is a name, then '$a^{k'}$' is a name.
4. Names are only those signs generated according to 1.-3.

variables:

1. 'x', 'y', 'z'.
2. If v is a variable, then v_1.
3. If v_i is a variable, then v_{i+1}.
4. Variables are only those signs generated according to 1.-3.

predicates:

1. 'F', 'G', 'H', 'I', 'J'.
2. 'F_1'.
3. If 'F_i' is a predicate, then 'F_{i+1}'.
4. Predicates are only those signs generated according to 1.-3.

poles: 'a', 'b'.

Auxiliary signs: '$-$', ','.

Regarding the use of forks, we lay down that a succession of n closed / n open forks is replaced by a closed / open fork with $n + 1$ prongs.

Contrary to the alphabet of Q, sentential connectives are not part of the alphabet of the ab-notation. Instead, forks and poles, 'a' and 'b', respectively, are part of the ab-notation, but not of *wff*s. Besides these differences, the alphabets of Q and that of the ab-notation are identical. We use 'a' and 'b' not only as poles, but also as names because they are commonly used as names. This does not lead to ambiguities because poles are flanked by '$-$' and occur between the prefix and propositional functions of ab-expressions, whereas names occur in the prefix of ab-expressions.

ab-expressions are unordered enumerations of "a-pole-groups" and "b-pole-groups", which, in turn, are unordered enumerations of "complex poles".

Pole-group: A pole-group is an unordered enumeration of complex poles in brackets. The complex poles are separated by commas.

a**-pole-groups:** a-pole-groups are concatenations of signs that begin with the pole 'a', followed by the hyphen '$-$', and ending with a pole-group.

b**-pole-groups:** b-pole-groups are concatenations of signs that begin with the pole 'b', followed by the hyphen '$-$', and ending with a pole-group.

Finally, we define the terminus "complex pole" syntactically. We base the definition of complex poles upon the definition of "elementary poles", i.e., complex poles of elementary predicate logic. The following sequence of definitions defines the terminus "elementary pole".

elementary sequence of names: An elementary sequence of names is an enumeration of names that are succeeded by a closed fork. The prongs of the closed forks connect the numbers of argument positions of one propositional function in the same line. The names in this sequence are separated by commas.

elementary sequence of existential quantifiers: An elementary sequence of existential quantifiers is an enumeration of existential quantifiers that are succeeded by a closed fork. The prongs of the closed forks connect the numbers of argument positions of one propositional function in the same line. The existential quantifiers in this sequence are separated by commas.

elementary sequence of universal quantifiers: An elementary sequence of universal quantifiers is an enumeration of universal quantifiers succeeded by a closed fork. The prongs of the closed forks connect the numbers of argument positions of one propositional function in the same line. The universal quantifiers in this sequence are separated by commas.

elementary sequence of quantifiers: An elementary sequence of quantifiers is a sequence of elementary sequences of existential and universal quantifiers. The elementary sequences of existential and universal quantifiers are not separated by commas in the elementary sequence of quantifiers.

elementary prefix: An elementary prefix is an expression composed of an elementary sequence of names followed by an elementary sequence of quantifiers. These two sequences are separated by a comma.

elementary suffix: An elementary suffix is an expression composed of a pole 'a' or 'b' flanked by hyphens, i.e., '$-a-$' or '$-b-$', followed by a predicate and a sequence of numbers $1 \ldots n$.

elementary pole: An elementary pole is an expression composed of an elementary prefix followed by an elementary suffix such that all numbers of the suffix occur once and only once in the prefix.

Elementary poles are single-lined.

We allow for elementary poles with a prefix composed of 0 names and 0 quantifiers. The suffix then consists of a predicate letter with 0 argument positions, i.e. a propositional variable. In this case, the elementary poles are propositional poles.

EXAMPLE: $b_1,\exists < {}^{2}_{4},\exists_5\forall_3 - b - F_{12345}$ is an elementary pole with prefix $b_1,\exists < {}^{2}_{4},\exists_5\forall_3$ and suffix $-b - F_{12345}$. The prefix is composed of the elementary sequence of names b_1 and the elementary sequence of quantifiers $\exists < {}^{2}_{4},\exists_5\forall_3$.

Unlike elementary poles, complex poles of molecular predicate logic have two dimensions. They can consist of several lines and columns. Lines are distinguished by propositional functions. If a complex pole contains n propositional functions, then it consists of n lines. With regard to columns of complex poles, we establish that they are counted from *right to left*, beginning with the column of propositional functions. Thus, the columns of propositional functions is column 1. Each successive column contains quantifiers or names – at most one quantifier or one name of a line.

To prepare the definition of complex poles, we establish that two existential quantifiers are "merged by a closed fork" by replacing them with one existential quantifier and connecting their forks using a closed fork. The same holds true in the case of merging two occurrences of the same name. Two universal quantifiers are "merged by an open fork" by replacing them with one universal quantifier and connecting their forks using an open fork.

EXAMPLE: $\exists_1$ and $\exists_1$ are merged to $\exists < {}^{1}_{1}$, $\forall < {}^{1}_{3}$ and $\forall_4$ are merged to $\forall <\substack{1\\3\\4}$.

Furthermore, we must distinguish between propositional functions of different lines[3] as "connected by an open fork" and as "connected by a closed fork".

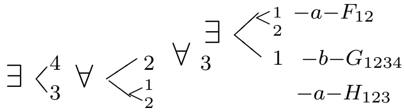
propositional functions connected by an open fork:

1. Propositional functions are connected by an open fork of column k if their lines are connected by an open fork of column k.
2. If a propositional function $\mathcal{A}_1$ is connected with a propositional function $\mathcal{A}_2$ by an open or closed fork of column i ($i < k$), and if $\mathcal{A}_2$ is connected with a propositional function $\mathcal{A}_3$ by an open fork of column k, then $\mathcal{A}_1$ and $\mathcal{A}_3$ are connected by an open fork of column k.
3. Propositional functions are connected by an open fork iff they are connected by an open fork according to 1. or 2.

propositional functions connected by a closed fork:

1. Propositional functions are connected by a closed fork of column k if (i) they are not connected by an open fork of column i ($i < k$) and (ii) their lines are connected by a closed fork of column k.
2. If a propositional function $\mathcal{A}_1$ is connected with a propositional function $\mathcal{A}_2$ by an open or closed fork of column i ($i < k$), and if $\mathcal{A}_2$ is connected with a propositional function $\mathcal{A}_3$ by a closed fork of column k, then $\mathcal{A}_1$ and $\mathcal{A}_3$ are connected by a closed fork of column k.
3. Propositional functions are connected by a closed fork iff they are connected by a closed fork according to 1. or 2.

EXAMPLE: In the following pole



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[3] If the same propositional function with the same innermost pole occurs in two lines, it can still be connected by an open or a closed fork. Thus, *occurrences* or *tokens* of propositional functions are connected. However, in the following we speak of "propositional functions connected by an open / a closed fork" for short.

the propositional functions of lines 1 and 2 are connected by a closed fork. The propositional functions of line 2 and 3, as well as of lines 1 and 3, are connected by an open fork.

By referring to the defined terms, complex poles can be defined syntactically as follows.

complex pole:

1. Every elementary pole is a complex pole.
2. Complex poles are generated in the following cases:
   (a) Given two complex poles with sequences of quantifiers preceded by a sequence of m and n ($m \leq n$) existential quantifiers, write one complex pole on the top of the other. Merge k ($k \leq m$) pairs of those existential quantifiers using a closed fork, and let the resulting sequence of existential quantifiers precede the sequence of quantifiers. Merge occurrences of identical names using a closed fork and write down all names in different columns in a sequence of names preceding the resulting complex pole.
   (b) Given two complex poles with sequences of quantifiers preceded by a sequence of m and n ($m \leq n$) universal quantifiers, write one complex pole on top of the other. Merge k ($k \leq m$) pairs of those universal quantifiers using an open fork, and let the resulting universal quantifiers precede the sequence of quantifiers. Merge occurrences of identical names using a closed fork, and write down all names in different columns in a sequence of names preceding the resulting complex pole.
3. Complex poles are generated in the following cases:
   (a) Given a complex pole with a universal quantifier preceding the sequence of quantifiers, and with m names in the sequence of names, replace k ($k \leq m$) of the m names by k existential quantifiers. Let the resulting sequence of k existential quantifiers precede the sequence of quantifiers in the prefix.
   (b) Given a complex pole with an existential quantifier preceding the sequence of quantifiers, and with m names in the sequence of names, replace k ($k \leq m$) of the m names by k universal quantifiers. Replace

closed forks of these k universal quantifiers by a succession of closed and open forks according to the following rules:

i. Connect numbers of the same line with closed forks.

ii. Connect numbers with a closed / open fork if the respective propositional functions are connected by a closed / open fork of column i but not of a column $i-k$.

Let the resulting sequence of k universal quantifiers precede the sequence of quantifiers in the prefix.[4]

4. Complex poles are only those poles generated according 1.-3.

EXAMPLE:

$$\begin{array}{llll} & {}^{4}\!\diagup\ \exists_1 & \exists\,\diagup^{2} & ,\exists_3\ -b-F_{1234}\\ \forall & & \quad\diagdown_{2} & \forall_3\ \exists_4\ -a-G_{1234}\\ & \diagdown_{2}\quad \forall\begin{array}{l}\diagup^{1}\\ \diagdown_{1}\end{array} & & -a-H_{12}\end{array}$$

is a complex pole, because

1. According to *rule 1* c_4,a_1,$\exists_2$,$\exists_3-b-F_{1234}$ and b_1,$\exists_2\forall_3\exists_4-a-G_{1234}$ and c_2,$\forall_1-a-H_{12}$ are complex poles.
2. According to *rule 2(a)* $$\begin{array}{lll} c_4,\ a_1, & \exists\,\diagup^{2} & ,\exists_3\ -b-F_{1234}\\ b_1, & \quad\diagdown_{2} & \forall_3\ \exists_4\ -a-G_{1234}\end{array}$$ is a complex pole.
3. According to *rule 3(b)* $$\begin{array}{lll} c_4,\ a_1, & \exists\,\diagup^{2} & ,\exists_3\ -b-F_{1234}\\ \forall_1 & \quad\diagdown_{2} & \forall_3\ \exists_4\ -a-G_{1234}\end{array}$$ is a complex pole.
4. According to *rule 2(b)* $$a_1,\ \begin{array}{llll} & {}^{4}\!\diagup & \exists\,\diagup^{2} & ,\exists_3\ -b-F_{1234}\\ c & ,\ \forall\begin{array}{l}\diagup^{1}\\ \diagdown_{1}\end{array} & \quad\diagdown_{2} & \forall_3\ \exists_4\ -a-G_{1234}\\ & \diagdown_{2} & & -a-H_{12}\end{array}$$ is a complex pole.
5. According to *rule 3(b)* $$a_1,\ \begin{array}{llll} & {}^{4}\!\diagup & \exists\,\diagup^{2} & ,\exists_3\ -b-F_{1234}\\ \forall & \quad\forall\begin{array}{l}\diagup^{1}\\ \diagdown_{1}\end{array} & \quad\diagdown_{2} & \forall_3\ \exists_4\ -a-G_{1234}\\ & \diagdown_{2} & & -a-H_{12}\end{array}$$ is a complex pole.

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[4] If, in consequence of these two rules, a comma directly succeeds another comma in the sequence of names, then one comma must be eliminated. If, as a consequence of these two rules, all names are replaced by quantifiers, then the comma preceding the sequence of quantifiers must be eliminated.

6. According to *rule 3(a)*
$$\begin{array}{lllllllll}
\exists_1 & & \diagup^{4} & & \exists\,\Big\langle^{2}_{2} & & ,\exists_3 & -b-F_{1234}\\
& \forall & & \forall\,\Big\langle^{1}_{1} & & \forall_3 & \exists_4 & -a-G_{1234}\\
& & \diagdown_{2} & & & & & -a-H_{12}
\end{array}$$
is a complex pole.

6.1.3.2 Translation of pole-groups into A_T and A_F

The a-pole-groups and the b-pole-groups can be translated into disjunctions of conjunctions of closed structures ($\bigvee \bigwedge cs$), respectively. $\bigvee \bigwedge cs$, to which a-pole-groups are translatable, we denote by A_T. A_T are equivalent to A. $\bigvee \bigwedge cs$, to which b-pole-groups are translatable, we denote by A_F. They are equivalent to $\neg A$.

To define the translation of pole-groups into A_T and A_F, we specify how complex poles are translated into closed structures.

complex poles $\Rightarrow$ closed structures:

1. Eliminate "$-a-$", and replace "$-b-$" by "$\neg$".

2. Replace all those numbers at argument positions of the propositional functions that occur in the sequence of names in the same line by the appropriate name. Finally, eliminate the sequence of names.

3. Replace all the remaining numbers with variables; replace all numbers connected by the prongs of a fork by the same variable, and numbers that are not connected by the prongs of a fork by different variables. Replace a number, n, in the propositional functions with the variable that replaces n in the same line in the sequence of quantifiers. Finally, eliminate the variables succeeding the prongs of forks, and write them down once between the respective quantifier and the respective fork.

4. Eliminate all forks except the innermost forks that connect different lines, and eliminate all commas.

5. The remaining expression is a two-dimensional representation of the hierarchy of a formula. This must be converted to a linear *wff*, proceeding from the inside to the outside by replacing open forks with disjunctions and closed forks with conjunctions. Disjunctions and conjunctions succeeding quantifiers need to be written in parentheses.

EXAMPLE: The pole $a_1,\ \forall \left\langle\begin{array}{l} ^4 \\ _1 \end{array}\right. \exists \left\langle\begin{array}{lll} ^2 & ,\exists_3 & -b-F_{1234} \\ _2 & \forall_3\ \exists_4 & -a-G_{1234} \end{array}\right.$ is to be translated in the closed structure

$$\forall x_1 \exists x_2 (\exists x_3 \neg F a x_2 x_3 x_1 \wedge \forall x_4 \exists x_5 G x_1 x_2 x_4 x_5),$$

because

1. Applying *rule 1* results in: $a_1,\ \forall \left\langle\begin{array}{l} ^4 \\ _1 \end{array}\right. \exists \left\langle\begin{array}{lll} ^2 & ,\exists_3 & \neg F_{1234}. \\ _2 & \forall_3\ \exists_4 & G_{1234} \end{array}\right.$
2. Applying *rule 2* results in: $\forall \left\langle\begin{array}{l} ^4 \\ _1 \end{array}\right. \exists \left\langle\begin{array}{lll} ^2 & ,\exists_3 & \neg F a_{234}. \\ _2 & \forall_3\ \exists_4 & G_{1234} \end{array}\right.$
3. Applying *rule 3* results in: $\forall x_1 \left\langle \exists x_2 \left\langle\begin{array}{ll} ,\exists x_3 & \neg F a x_2 x_3 x_1. \\ \forall x_4\ \exists x_5 & G x_1 x_2 x_4 x_5 \end{array}\right.\right.$
4. Applying *rule 4* results in: $\forall x_1 \quad \exists x_2 \left\langle\begin{array}{ll} ,\exists x_3 & \neg F a x_2 x_3 x_1. \\ \forall x_4\ \exists x_5 & G x_1 x_2 x_4 x_5 \end{array}\right.$
5. Applying *rule 5* results in:

$$\forall x_1 \exists x_2 (\exists x_3 \neg F a x_2 x_3 x_1 \wedge \forall x_4 \exists x_5 G x_1 x_2 x_4 x_5).$$

Given the translation of complex poles to closed structures, the translation of pole-groups to A_T and A_F is straightforward.

pole-groups $\Rightarrow$ A_T and A_F:

1. Translate a-pole-groups to A_T and b-pole-groups to A_F.
   (a) Connect the translation of single complex poles by "$\wedge$".
   (b) Connect the translations of single pole-groups by "$\vee$".

6.1.3.3 Translation of A_T and A_F into pole-groups

To define how to translate A_T and A_F into pole-groups, we must specify how closed structures are translated into complex poles.

closed structures $\Rightarrow$ complex poles: Translate a closed structure A, moving from the inside to the outside according to the hierarchy of A, by applying the following rules:

1. Translate negated propositional functions $\neg\mathcal{A}$ into $-b - \mathcal{A}$, and translate non-negated propositional functions $\mathcal{A}$ into $-a - \mathcal{A}$.
2. In the case of conjunctions / disjunctions, write down the translations of the conjuncts / disjuncts, one on top of the other.
3. Write a quantifier in a new column to the left. Replace its variable in the propositional functions with the number of the argument position at which the variable occurs. If an occurrence of the variable is replaced by the number n in the m-th line, then write down n in the m-th line behind the quantifier in the prefix.
4. (a) Connect numbers succeeding an existential quantifier with closed forks.
   (b) Numbers succeeding a universal quantifier must be connected by the following rules:
      i. Connect numbers of the same line with closed forks.
      ii. Connect numbers with a closed / open fork if the respective propositional functions are connected with a closed / open fork of column i, but not of a column $i - k$ (cf. p. 239).
      iii. Finally, connect numbers not yet connected with open forks.
5. Replace a name in the propositional functions with the number of the argument position at which the name occurs. Furthermore, incorporate the name in the sequence of names that precedes the sequence of quantifiers in the prefix of the constructed complex pole. If an occurrence of the name is replaced with the number n in the m-th line, then write down n in the m-th line behind the name in the prefix. Connect the numbers behind the name in the prefix with closed forks.
6. Separate individual items in sequences of existential quantifiers, universal quantifiers, and names using commas. Separate the sequence of names from the sequence of quantifiers with a comma.

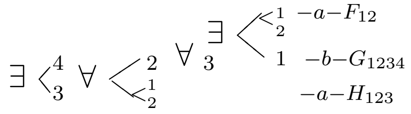
7. A closed structure A is translated in a complex pole iff the outermost quantifier and all names of A are considered.

EXAMPLE:

The closed structure

$$\exists y_1 \forall x_2 (\forall x_1 \exists y_2 (F y_2 y_2 \wedge \neg G y_2 x_2 x_1 y_1) \vee H x_2 x_2 y_1)$$

is to be translated in the complex pole



because

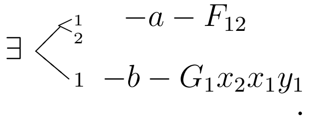
1. Applying *rule 1* results in:

$$-a - F y_2 y_2, \;\; -b - G y_2 x_2 x_1 y_1.$$

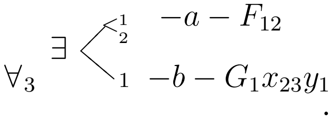
2. Applying *rule 2* results in:

$$\begin{array}{c} -a - F y_2 y_2 \\ -b - G y_2 x_2 x_1 y_1 \end{array} .$$

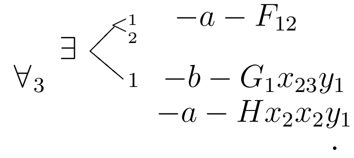
3. Applying *rule 3* and *rule 4(a)* results in:



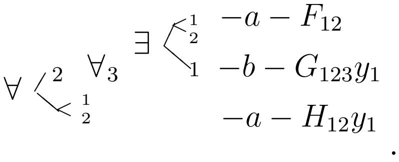
4. Applying *rule 3* results in:



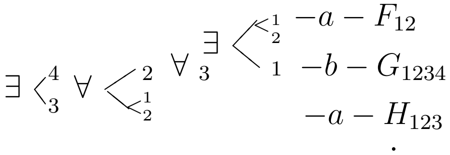
5. Applying *rule 1* and *rule 2* results in:



6. Applying *rule 3* and *rule 4(b)* results in:



7. Applying *rule 3* and *rule 4(a)* results in:



8. According to *rule 7* the resulting complex pole is the translation of the initial closed structure.

Given the translation of closed structures into complex poles, the translation of A_T and A_F into pole-groups is straightforward.

A_T**,** $A_F \Rightarrow$ **pole-groups:** Write all translations of closed structures of a disjunct of A_T / A_F in an a-pole-group / b-pole-group. Separate the complex poles of one pole-group and the a-pole-groups / b-pole-groups by commas.

6.2 $\bigvee\bigwedge cs$ and pole-groups

This section defines a procedure to convert *wff*s to pole-groups. One possible way to perform this conversion is analogous to the procedure in elementary predicate logic. First, *wff*s are converted to ab-diagrams, and then ab-diagrams are converted to pole-groups. The first step is already sufficiently defined for the whole realm of predicate logic on p. 121. For the second step, the *path-rule*, p. 158 must be replaced by a more general rule. However, on the basis of the definitions of the preceding section, another method for converting *wff*s to pole-groups is possible. Such a method first defines a procedure to convert *wff*s to A_T and A_F, and then translates A_T and A_F into a- and b-pole-groups. This procedure is defined in this section. It has two advantages compared to the other procedure. First and foremost, one can make use of known equivalence rules. This demonstrates that the procedure defines an equivalence transformation. Second, this method has the practical advantage of dispensing with the complex graphical construction of ab-diagrams.

The concept of disjunctive normal forms in the realm of predicate logic is not new. The notion of inverting the procedure for generating prenex normal forms is also the main idea of Hintikka's "distributive normal forms" (Hintikka (1973), cf. also Oglesby (1962) and Stegmüller (1984), p. 241). However, Hintikka works out his distributive normal forms in the framework of his theory of information. Because of this, his distributed normal forms do not imply strategies of minimization. Instead, they are a kind of canonical disjunctive normal form in predicate logic. Unlike Hintikka's distributive normal forms, the disjunctions of conjunctions of closed structures primarily serve to solve the equivalence problem, the problem of semantics and the problem of implication. They are part of the realization of Wittgenstein's *New Logic*, including *New Semantics*, which is based on the concept of closed structures. By implementing minimization strategies, they are not as unmanageable as Hintikka's distributed normal forms. Furthermore, Hintikka does not develop the idea of a maximal minimization of the quantifier's scope to the extent of the procedure specified in this section.[5]

In addition to the assumption that we refer to NNFs only (cf. p. 226), we establish that each variable occurring in the resulting NNFs is bound by only one quantifier. This is established in order to identify single quantifiers by the shape of the variable which it binds. This assumption can be satisfied by applying SUB1 and SUB2, cf. the list of rules on p. 337. The following *wff*, e.g., does not satisfy this stipulation:

$$\exists x Fx \wedge \forall x Gx \tag{6.13}$$

However, by applying SUB2, (6.13) can be converted to $\exists x Fx \wedge \forall y Gy$, thus satisfying the aforementioned presumption. Thus, if we refer in the following to

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[5] In particular, this applies to Hintikka's "distributive normal forms of the second kind", cf. Hintikka (1970), p. 266.

NNFs, we refer to NNFs with each variable only bound by one quantifier. NNFs of this kind are considered as *wff*s which are to be converted to $\bigvee\bigwedge cs$. Thus, given a *wff* A, one first converts A and $\neg A$ to NNFs of the described kind. The $\bigvee\bigwedge cs\ A_T$ is then generated from the NNF of A, and the $\bigvee\bigwedge cs\ A_F$ is generated from the NNF of $\neg A$.

The objective of converting a *wff* A to a $\bigvee\bigwedge cs$ is to apply PN-laws in order to minimize scope of quantifiers to a maximal extent. If a bound variable does not occur in all conjuncts / disjuncts of a quantifier's scope, the scope of the respective quantifier can be minimized by PN1-8. If necessary, the application of these laws is prepared by the associative and the commutative laws, ASS$\wedge$, ASS$\vee$, COM$\wedge$, COM$\vee$. However, the situation is different if the bound variable occurs in both conjuncts / disjuncts of a quantifier's scope. In this case, only the following two PN-laws are available in terms of rules of equivalence:

$$\begin{array}{lcll} \forall\nu(A(\nu)\wedge B(\nu)) & \dashv\vdash & \forall\nu A(\nu)\wedge\forall\nu B(\nu) & \text{PN9} \\ \exists\nu(A(\nu)\vee B(\nu)) & \dashv\vdash & \exists\nu A(\nu)\vee\exists\nu B(\nu) & \text{PN10} \end{array}$$

If the scope of a universal quantifier is a disjunction, or if the scope of the existential quantifier is a conjunction and the bound variable occurs in both disjuncts / conjuncts, then the scope of the respective quantifiers cannot be minimized any further by an equivalence PN-law. To guarantee a maximal minimization of the scopes, and also to generate disjunctions of conjunctions of *closed structures*, such disjunctions / conjunctions must be converted to conjunctive / disjunctive normal forms.[6] For example, the following *wff*

$$\forall x((Fx\wedge Gx)\vee(Hx\wedge Ix)) \tag{6.14}$$

is not a closed structure, and no PN-law is applicable to (6.14). However, the scope of the universal quantifier can be minimized further, and the whole formula can be converted to a $\bigvee\bigwedge cs$ by converting the scope to a CNF:

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[6] By "conjunctive normal forms" we mean conjunctions of disjunctions. If one replaces the respective disjuncts by propositional variables one receives CNFs of propositional logic. By "disjunctive normal forms" we mean disjunctions of conjunctions. Replacing the respective conjuncts by propositional variables results in DNFs of propositional logic. Thus, if we speak of CNFs and DNFs we treat the respective conjuncts and disjuncts like propositional variables. The same holds in case of canonical conjunctive and disjunctive forms, CCNFs and CDNFs, as well as in case of the reduced conjunctive and disjunctive normal forms, RCNFs and RDNFs.

$$\forall x((Fx \vee Hx) \wedge (Fx \vee Ix) \wedge (Gx \vee Hx) \wedge (Gx \vee Ix)). \qquad (6.15)$$

Contrary to (6.14), PN9 is applicable to (6.15). By applying PN9 several times, one yields the formula (6.16), which is a conjunction of cs:

$$\forall x(Fx \vee Hx) \wedge \forall x(Fx \vee Ix) \wedge \forall x(Gx \vee Hx) \wedge \forall x(Gx \vee Ix). \qquad (6.16)$$

Analogously, the scope of an existential quantifier must be converted to a disjunctive normal form DNF if the variable bound by the existential quantifier occurs in both conjuncts.

More precisely, reduced conjunctive normal forms (RCNFs) and reduced disjunctive normal forms (RDNFs) must be generated from the canonical conjunctive and disjunctive normal forms (CCNFs and CDNFs), both to solve the equivalence problem and to avoid unnecessarily complex CNFs and DNFs. Only generating RCNFs and RDNFs guarantees unambiguous minimization strategies, cf. p. 208. For this reason, we will make use of an unambiguous *propositional* minimization strategy according to the Quine-McCluskey algorithm. A RCNF is generated by the Quine-McCluskey algorithm by first generating the canonical conjunctive normal form CCNF. This CCNF must be negated. The resulting ¬CCNF must be converted to a CDNF by applying DM$\wedge$ and DM$\vee$. Then, the Quine-McCluskey algorithm according CDNFs is applied to the respective CDNF. By negating the resulting RDNF and applying DM$\vee$ and DM$\wedge$ again one finally receives the RCNF. Negation signs will only occur left to propositional functions as they only occured left to propositional functions before converting the initial formula to a CCNF. The minimization strategies that generate RCNFs and RDNFs must be carried out after generating a CCNF or a CDNF, but before applying PN9 and PN10. The respective RCNFs and RDNFs are unique with respect to propositional logic. That is, they are unique insofar as one abstains from internal relations not reducible to propositional logic.

RCNF-R.:

1. If the variable bound by the universal quantifier occurs in both disjuncts, convert a disjunction in the scope of a universal quantifier to its CCNF.

2. Reduce the resulting CCNF to its RCNF according to the Quine-McCluskey algorithm.[7]

RDNF-R.:

1. If the variable bound by the existential quantifier occurs in both conjuncts, convert a conjunction in the scope of an existential quantifier to its CDNF.
2. Reduce the resulting CDNF to its RDNF according to the Quine-McCluskey algorithm.[8]

Beyond applying associative and commutative laws and reducing CCNFs and CDNFs to RCNFs and RDNFs, the application of PN-laws must be prepared according to the following rules, if necessary:

$$\begin{array}{lcll} \forall\mu\forall\nu A(\mu,\nu) & \dashv\vdash & \forall\nu\forall\mu A(\mu,\nu) & \forall Ex. \\ \exists\mu\exists\nu A(\mu,\nu) & \dashv\vdash & \exists\nu\exists\mu A(\mu,\nu) & \exists Ex. \end{array}$$

The following formula (6.17), for example, is not a closed structure because the scope of the preceding existential quantifier is not completely minimized.

$$\exists x\exists y(Fxy \land Gy). \tag{6.17}$$

In order to apply PN6, one must first apply $\exists Ex.$.

Before defining the minimization rule of the scope of universal and existential quantifiers, we must introduce two additional termini:

μ-disjunction: A disjunction with disjuncts that are not conjunctions, and all disjuncts containing the variable μ, which is bound by a universal quantifier, is a μ-disjunction.

μ-conjunction: A conjunction with conjuncts that are not disjunctions, and all conjuncts containing the variable μ, which is bound by an existential quantifier, is a μ-conjunction.

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[7] We presume that this implies eliminating tautologous conjuncts, i.e., disjunctions containing A and $\neg A$ as parts of proper conjunctions.

[8] We presume that this implies the elimination of contradictory disjuncts, i.e., conjunctions containing A and $\neg A$ as parts of proper disjunctions.

To minimize the scope of universal and existential quantifiers by applying PN-laws maximally, the $\forall$-rule and the $\exists$-rule must be applied:

$\forall$-R.: minimize the scope of $\forall\mu$, by applying PN1-4, PN9 and $\forall Ex.$, until all occurrences of $\forall\mu$ are directly to the left of either an existential quantifier, a negation sign, a predicate letter, or a μ-disjunction. Apply PN1-4 before applying PN9. Preparation by COM$\wedge$, ASS$\wedge$, COM$\vee$, ASS$\vee$, $\forall Ex.$, in order to isolate disjuncts and conjuncts not containing μ may be necessary. If the scope of $\forall\mu$ is a disjunction after having applied PN1-4, apply RCNF-R.

$\exists$-R.: minimize the scope of $\exists\mu$ by applying PN5-8, PN10, and $\exists Ex.$ until all occurrences of $\exists\mu$ are either directly to the left of an universal quantifier, a negation sign, a predicate letter or a μ-conjunction. Apply PN5-8 before applying PN10. Preparation by COM$\wedge$, ASS$\wedge$, COM$\vee$, ASS$\vee$, $\exists Ex.$ in order to isolate disjuncts and conjuncts not containing μ may be necessary. If the scope of $\exists\mu$ is a conjunction after having applied PN1-4, apply RDNF-R.

By referring to these two rules, the procedure of converting an NNF B to an equivalent $\bigvee\bigwedge cs$ can be defined as follows:

$\bigvee\bigwedge cs$-R.: convert an NNF B to an equivalent $\bigvee\bigwedge cs$ by applying $\exists$-R. to each existential quantifier, and $\forall$-R. to each universal quantifier of B, iteratively moving from the inside to the outside according to the hierarchy of B. Finally, apply RDNF to generate a resulting $\bigvee\bigwedge cs$.

By applying $\bigvee\bigwedge cs$-R., an NNF B is converted to an *equivalent* $\bigvee\bigwedge cs$, since only rules of equivalence are applied. The procedure terminates because, as every NNF B has only a finite number of quantifiers, by moving from the inside to the outside, one will eventually end up with the outermost quantifier. Furthermore, as every formula has only a finite number of conjunctions, disjunctions, and quantifiers, applying the rules defining $\exists$-R. and $\forall$-R., as well as the final application of RDNF-R., will bring the procedure to an end. Per the definition of $\exists$-R., $\forall$-R. and RDNF-R., the result will be a $\bigvee\bigwedge cs$. Due to the definition of RCNF-R. and RDNF-R., the resulting $\bigvee\bigwedge cs$ are unique with respect to propositional logic.

To convert a *wff* to an ab-expression, the following rule must be applied:

wff $\Rightarrow$ ab-expression:

1. Generate the NNFs of A and $\neg A$.
2. Convert A and $\neg A$ to A_T and A_F by applying $\bigvee\bigwedge cs$-R.
3. Translate A_T into a-pole-groups, and translate A_F into b-pole-groups.

EXAMPLE:

$$\exists x_1 \forall x_2 (Gx_1 \wedge (\neg Fx_1x_2 \vee \forall x_4 \neg Fx_4x_4)) \wedge \exists x_5 Fx_5x_5 \tag{6.18}$$

This formula is an NNF. In the following, we exemplify how to convert (6.18) to A_T by applying $\bigvee\bigwedge cs$-R.

First, one applies $\forall$-R. to $\forall x_4$ and $\exists$-R. to $\exists x_5$. In this case, the formula will not change because $\forall x_4$ is directly to the left of a negation sign, and $\exists x_5$ is directly to the left of a predicate letter. One then applies $\forall$-R. to $\forall x_2$ in (6.18). Applying PN1 or PN4 leads to:

$$\exists x_1 (Gx_1 \wedge (\forall x_2 \neg Fx_1x_2 \vee \forall x_4 \neg Fx_4x_4)) \wedge \exists x_5 Fx_5x_5. \tag{6.19}$$

Next, one applies $\exists$-R. to $\exists x_1$ in (6.19). The scope of $\exists x_1$ is not a μ-conjunction, because the second conjunct is a disjunction. According to $\exists$-R., one must apply RDNF-R. in order to apply PN10, and then PN6. One obtains the following result:

$$(\exists x_1 (Gx_1 \wedge \forall x_2 \neg Fx_1x_2) \vee (\exists x_1 Gx_1 \wedge \forall x_4 \neg Fx_4x_4)) \wedge \exists x_5 Fx_5x_5. \tag{6.20}$$

A final application of RDNF-R. yields the $\bigvee\bigwedge cs$:

$$\begin{gathered} \exists x_1 (Gx_1 \wedge \forall x_2 \neg Fx_1x_2) \wedge \exists x_5 Fx_5x_5 \quad \vee \\ \exists x_1 Gx_1 \wedge \forall x_4 \neg Fx_4x_4 \wedge \exists x_5 Fx_5x_5. \end{gathered} \tag{6.21}$$

Translation of this $\bigvee\bigwedge cs$ results in the a-pole-groups:

$$a - \{\exists < {}^{1}_{1}\forall_{2}{}^{-a-G_1}_{-b-F_{12}}, \exists < {}^{1}_{2} - a - F_{12}\},$$

$$a - \{\exists_1 - a - G_1, \forall <^{1}_{2} -b - F_{12}, \exists <^{1}_{2} -a - F_{12}\}.$$

6.3 New Semantics

This section elaborates alternative semantics for predicate logic, which is based on reducing predicate logic to $\bigvee \bigwedge cs$ and translating it into ab-expressions. In classical semantics, single interpretations $\Im(A)$ of a *wff* A can only be evaluated one by one, in order to identify whether the respective $\Im(A)$ is a model or a counter-model. In contrast, New Semantics makes it possible to construct the class of models and counter-models without evaluating single interpretations. The reason for this is that ab-expressions serve as identity criteria of both models and counter-models. Thus, they can be used as rules for the systematic construction of the class of models and counter-models. Thus, given a *wff* A, one can generate its ab-expression in a finite number of steps and then construct the models and counter-models from the ab-expression, without deciding whether each interpretation is a model or counter-model individually. Before explaining this conception of New Semantics in section 6.3.3–6.3.5 in detail, we define classical semantics in section 6.3.1, without referring to any "standard interpretation" in terms of ordinary language expressions. We then contrast the two conceptions – classical semantics and New Semantics – in section 6.3.2.

6.3.1 Classical semantics

By classical semantics, we mean the way to interpret *wff*s according to traditional logic and how to distinguish their interpretations in models and counter-models. In the following, we first define the totality of interpretations of predicate logic Q, i.e. $\Im(Q)$. Then, we define how one divides $\Im(Q)$ in the class of models of a *wff* A, i.e. $\Im_T$ of A, and the class of counter-models of A, i.e. $\Im_F$ of A.

6.3.1.1 $\Im_Q$

– As it is abstained from any specified meaning within New Logic, such a construction of possible extensions can only be based on a unspecified infinite domain $\{c_1, c_2, \ldots\}$ (plus finite parts $\{c_1, \ldots c_n\}$ of it and besides unspecified "values" a and b instead of T and F). However, due to the Löwenheim-Skolem theorem this also suffices from the traditional point of view to specify logically true formulas. –

We define the semantics of Q, without interpreting names and predicates by means of ordinary language expressions with a certain meaning. Rather we solely refer to objects that are specified as objects $c_1, c_2, \ldots$. On this basis, we can define both the objects in terms of values of a domain I, as well as possible domains I, inductively. According to Wittgenstein's terminology, we define them in terms of operations.

values of I: A value of I is defined as follows:

1. c_1 is a value.
2. If c_i is a value, then c_{i+1} is also a value.
3. Values of I are only values according to 1. or 2.

domain I: A domain I is defined as follows:

1. $\{c_1\}$ and $\{c_1, c_2\}$ are domains I.
2. If $\{c_1, \ldots c_i\}$ is a domain I, then $\{c_1, \ldots c_i, c_{i+1}\}$ is also a domain. Here "..." denotes an enumeration of values with indices > 1 and $< i$.
3. $\{c_1, c_2, \ldots\}$ is a domain I. Here "..." denotes an endless enumeration of values, in which every c_i is followed by c_{i+1}.
4. A domain I is only a domain according to 1.-3.

Instead of defining I directly, in terms of sets of natural numbers, we refer to natural numbers as indices of the letter 'c'. This is because we want to be able to replace names in *wff*s by signs of the form "c_i". We make use of such expressions in the definition of models (cf. p. 259). However, by replacing names with numerals, an expression such as "F11" is ambiguous. It could either result from replacing 'a' by '1' in Faa, or from replacing 'a' by '11' in Fa. On the other hand, referring to numbers as indices of the letter 'c' yields the corresponding, unambiguous results Fc_1c_1 and Fc_{11}.

In the following, we distinguish between interpreting some sign from the alphabet of Q, and the totality of interpretations of signs of a certain kind from the alphabet of Q. Thus, $\Im(\mathcal{J})$ denotes the interpretation of a propositional variable $\mathcal{J}$, whereas $\widehat{\Im(\mathcal{J})}$ stands for a totality of interpretations of all propositional variables. The same holds for $\Im(t)$ / $\widehat{\Im(t)}$ and $\Im(\varphi)$ / $\widehat{\Im(\varphi)}$ regarding the interpretations of names and predicates.

$\Im(\mathcal{J})$ is the value assigned to a propositional variable $\mathcal{J}$, that is, either the truth value T or the truth value F. $\Im(t)$ is the value assigned to a name t, which is some element c_i from domain I. $\Im(\varphi)$ of an n-ary predicate φ is a set of n-tuples from I. Variables are interpreted by $I = \Im(\mathrm{v}) = I$ (v is a meta-variable for the set of variables of Q). A single interpretation $\Im$ is defined as follows:

$\Im$: A single interpretation $\Im$ is an ordered enumeration of $\Im(\mathrm{v})$, $\widehat{\Im(\mathcal{J})}$, $\widehat{\Im(t)}$, $\widehat{\Im(\varphi)}$.

Every single interpretation $\Im$ begins with $\Im(\mathrm{v})$, followed by $\widehat{\Im(\mathcal{J})}$, succeeded by $\widehat{\Im(t)}$, and followed by $\widehat{\Im(\varphi)}$. In $\widehat{\Im(\mathcal{J})}$, $\widehat{\Im(t)}$, and $\widehat{\Im(\varphi)}$, the interpretations of propositional variables / names / predicates without indices precede interpretations of propositional variables / names / predicates with indices. The interpretations of signs with small indices precede interpretations with a larger index. In $\widehat{\Im(\varphi)}$, interpretations of n-ary predicates precede interpretations of $n+1$-ary predicates. Thus, each interpretation, $\Im$, has the following form:[9]

$\Im(\mathrm{v})$	$\widehat{\Im(\mathcal{J})}$	$\widehat{\Im(t)}$	$\widehat{\Im(\varphi)}$
$\Im(x,y,z,x_1,y_1,\ldots) = \{\ldots\}$	$\Im(P) = \ldots$	$\Im(a_1) = \ldots$	$\Im(F) = \{\ldots\}$
	$\Im(Q) = \ldots$	$\Im(a_2) = \ldots$	$\Im(G) = \{\ldots\}$
	$\Im(R) = \ldots$	$\Im(a_3) = \ldots$	$\Im(H) = \{\ldots\}$
	$\Im(S) = \ldots$	$\vdots$	$\Im(I) = \{\ldots\}$
	$\Im(T) = \ldots$		$\Im(J) = \{\ldots\}$
	$\Im(U) = \ldots$		$\Im(F_1) = \{\ldots\}$
	$\Im(P_1) = \ldots$		$\Im(F_2) = \{\ldots\}$
	$\Im(P_2) = \ldots$		$\Im(F_3) = \{\ldots\}$
	$\vdots$		$\vdots$
			$\Im(F) = \{\ldots\}$
			$\Im(G) = \{\ldots\}$
			$\Im(H) = \{\ldots\}$
			$\Im(I) = \{\ldots\}$

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[9] For the sake of brevity, we write $\Im(\mathrm{v})$, $\widehat{\Im(\mathcal{J})}$, $\widehat{\Im(t)}$, $\widehat{\Im(\varphi)}$ in different columns.

			$\Im(J) = \{\ldots\}$
			$\Im(F_1) = \{\ldots\}$
			$\Im(F_2) = \{\ldots\}$
			$\vdots$

The interpretation of Q is defined as follows:

$\Im(Q)$: $\Im(Q)$ is the totality of all single interpretations $\Im$.

As the number of propositional variables, names, and predicates of Q is infinite, $\widehat{\Im(\mathcal{J})}$, $\widehat{\Im(t)}$, and $\widehat{\Im(\varphi)}$ consist of infinite elements. As I may contain an infinite number of objects, $\Im(\varphi)$ may contain an infinite number of tuples, and $\Im(Q)$ consists of an infinite number of single $\Im$.

$\Im(Q)$ can be constructed systematically by methodically varying $\Im(\mathrm{v})$, $\Im(\mathcal{J})$, $\Im(t)$, and $\Im(\varphi)$.

1. I (= $\Im(\mathrm{v})$) can be varied systematically by starting with $c_i = c_1$ as single element of I, and then adding a further element c_{i+1} for another I. In addition, $I = \{c_1, c_2, \ldots\}$ – the domain with an unlimited number of objects – must be constructed.

2. For every I with i elements, a single $\Im(\mathcal{J})$, $\Im(t)$, and $\Im(\varphi)$ can be varied systematically:

   - the number of possible $\Im(\mathcal{J})$ is 2 because two truth values, T and F, can be assigned to each propositional variable.
   - the number of possible $\Im(t)$ is i.
   - the number of possible different k-tuples is i^k, and the number of possible $\Im(\varphi)$ of a k-ary predicate φ is $2^{(i^k)}$.

3. By combining the single variations of each $\Im(\mathcal{J})$, $\Im(t)$, and $\Im(\varphi)$, all interpretations $\Im$ for a given domain I can be constructed. Given u predicates with arity $k_1 \ldots k_u$, m names, and n propositional variables, one has

   $$2^n \cdot i^m \cdot 2^{(i^{k_1}+i^{k_2}+\ldots+i^{k_u})}$$

   interpretations $\Im$ for a given I with i elements.

$\Im(Q)$ is constructed methodically in the sense that one can generate new interpretations by systematically varying previous interpretations. According to Wittgenstein's terminology, it is possible to define $\Im(Q)$ by defining *operations* in terms of rules that describe how to vary interpretations systematically.

By the term "interpretations of a *wff* A", $\Im(A)$, we denote the totality of interpretations that only contain interpretations of variables, propositional variables, names, and predicates *occurring in* A. Correspondingly, by $\Im_T(A)$ and $\Im_F(A)$, we denote the class of models and the class of counter-models of A. $\Im(Q)$ is absolute, whereas $\Im(A)$ depends on the syntax of some *wff* A. Strictly speaking, the totality of models, $\Im_T$, and the totality of counter-models, $\Im_F$, of a *wff* A form divisions of $\Im(Q)$, and not only of $\Im(A)$. This is true by the following reasoning:

1. Let A and B be equivalent *wff*s that do not contain the same signs, for example, $P \vee \neg P$ and $Q \vee \neg Q$, or P and $P \wedge (Q \vee \neg Q)$ or $\forall x Fx$ and $\forall x(Fx \wedge (Gx \vee \neg Gx))$.
2. The following definition of equivalence is valid: A and B are equivalent iff they have the *same* models (and the same counter-models).
3. Therefore, A and B have the *same* models and the *same* counter-models.
4. $\Im_T(A) \neq \Im_T(B)$ and $\Im_F(A) \neq \Im_F(B)$, because A and B do not contain the same signs according to 1.
5. Thus, to satisfy the definition of equivalence in 2, one must rely on $\Im(Q)$ and its classification in $\Im_T$ and $\Im_F$, and not on $\Im(A)$ and its classification in $\Im_T(A)$ and $\Im_F(A)$.

However, the models and counter-models of a *wff* A do not depend on the interpretations of signs not contained in A. All interpretations of $\Im(Q)$ that are not part of $\Im(A)$ can vary arbitrarily, without influencing $\Im$'s being a model or counter-model. Thus, it suffices to refer to $\Im(A)$ to *identify* $\Im_T$ and $\Im_F$ of A. However, the *so-called* "models" and "counter-models" do not consist of $\Im(A)$ alone, but of I, $\widehat{\Im(\mathcal{J})}$, $\widehat{\Im(t)}$, and $\widehat{\Im(\varphi)}$.

6.3.1.2 $\Im_T$ and $\Im_F$

$\Im_T$: $\Im_T$ of a *wff* A is the totality of models of A.

$\Im_F$: $\Im_F$ of a *wff* A is the totality of counter-models of A.

$\Im(Q)$ is the set union of $\Im_T$ and $\Im_F$:

$$\Im(Q) = \Im_T \cup \Im_F$$

Each single $\Im$ is either a model of A, $\Im \models A$, or a counter-model of A, $\Im \not\models A$, but not both.

$$\Im_T \cap \Im_F = \{\}.$$

Thus, if $\Im_T$ is given, $\Im_F$ is given too, and vice versa.

To identify whether $\Im \models A$ holds, or when $\Im \not\models A$ holds, we introduce the following variables:

A: meta-variable for arbitrary *wff* of Q.

$\mathcal{J}$: meta-variable for propositional variables of Q.

φ: meta-variable for predicates of Q.

t: meta-variable for names of Q.

e: meta-variable for elements of I.

v: meta-variable for variables of Q.

$\mathcal{A}, \mathcal{B}$: meta-variable of arbitrary A, and of expressions that result from replacing one or more names of a *wff* A by elements of I.

$\mathcal{S}$: meta-variable for „open schemata". We call expressions resulting from eliminating the preceding quantifier of a *wff* A "open schemata." Thus, in an open schemata, there is one, and only one, free variable.

$\mathcal{A}$ is satisfied, for example, by P, Fa, $\exists x(Fx \vee Gx)$, $\forall xFx \vee Q$, $Fa \vee Fb$, $\exists x\forall y(Fxy \to Ga)$. It is also satisfied by Fc_1 (resulting, e.g., from Fa), $Fc_1 \vee Fc_2$ (resulting, for example, from $Fa \vee Fb$), $\forall x(Fc_1x \to Ga)$ (resulting, e.g., from $\forall x(Fbx \to Ga)$).

By $\varphi(t_1 \dots t_m/e_1 \dots e_n)$, we refer to expressions $\mathcal{A}$ that contain a predicate φ, m names $t_1 \dots t_m$, and n elements $e_1 \dots e_n$ from I ($m \geq 0, n \geq 0, m + n > 0$). For example, Fac_1, $Faab$, Fc_1c_2a, Fc_1c_1. By $\mathcal{S}\frac{e}{v}$, we refer to expressions $\mathcal{A}$ that result from replacing the free variable, v, in an open scheme $\mathcal{S}$ by an element e wherever v occurs in $\mathcal{S}$.

model: $\Im \models A$ iff $\Im \models A$ holds by applying *rules 1-8* from the inside to the outside.

1. $\Im \models \mathcal{J}$ iff $\Im(\mathcal{J}) = T$.
2. $\Im \models \varphi(t_1 \dots t_m/e_1 \dots e_n)$ iff $\Im(\varphi)$ applies to $\Im(t_1) \dots \Im(t_m)$ and $e_1 \dots e_n$.[10]
3. $\Im \models \neg\mathcal{A}$ iff $\Im \not\models \mathcal{A}$.
4. $\Im \models (\mathcal{A} \wedge \mathcal{B})$ iff $\Im \models \mathcal{A}$ and $\Im \models \mathcal{B}$.
5. $\Im \models (\mathcal{A} \vee \mathcal{B})$ iff $\Im \models \mathcal{A}$ or $\Im \models \mathcal{B}$.
6. $\Im \models \forall v\mathcal{S}$ iff for all e from I $\Im \models \mathcal{S}\frac{e}{v}$.
7. $\Im \models \exists v\mathcal{S}$ iff for some e from I $\Im \models \mathcal{S}\frac{e}{v}$.

Rules 2-5 refer to models of expressions $\mathcal{A}$ or $\mathcal{B}$ in general, and not only to *wff* A. This is because we intend to use a recursive definition when an interpretation $\Im$ satisfies a scheme $\mathcal{S}$. This makes it possible to base the application of *rule 6* and *rule 7*, which refer to elements e from I, on recursive definitions also.

In contrast to the definition of $\Im(Q)$, the definitions of $\Im_T$ and $\Im_F$ only depend on $\Im(\mathcal{J})$, $\Im(t)$, and $\Im(\varphi)$ of $\Im(A)$. From this, it does not follow that all remaining $\Im(\mathcal{J})$, $\Im(t)$, and $\Im(\varphi)$ are not part of $\Im_T$ and $\Im_F$. Instead, it only means that they have no influence on the identification of models and counter-models.

Based on the fact that it suffices to refer to a finite number of $\Im(\mathcal{J})$, $\Im(t)$, and $\Im(\varphi)$ by identifying the models of A, and the fact that *rules 1-7* can be applied recursively, it does not follow that it is possible to compute whether a single $\Im$ is a model of a *wff* A. Instead, this is only possible for a finite I.[11] A programme systematically constructing and computing finite interpretations of a *wff* A, one that is based on the definitions of this section, can be viewed at the following website:

http://philoscience.unibe.ch/logik.html (tools)

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[10] $\Im(\varphi)$ applies to $\Im(t_1) \dots \Im(t_m)$ and $e_1 \dots e_n$ iff there is a tuple in $\Im(\varphi)$ with $\Im(t_i)$ at those positions t_i occupies in $\varphi(t_1 \dots t_m/e_1 \dots e_n)$, and with e_j at those positions e_j occupies in $\varphi(t_1 \dots t_m/e_1 \dots e_n)$. Given $\varphi(t_1 \dots t_m/e_1 \dots e_n) = Fac_1bc_1$ and $\Im(a) = c_2$ and $\Im(b) = c_3$, then $\Im(\varphi)$ applies to $\Im(a)$, $\Im(b)$ and c_1 iff $\Im(\varphi)$ contains the 4-tuple (c_2, c_1, c_3, c_1).

[11] cf. Lampert (2005a), chapter 8.2-8.3 for a detailed discussion.

6.3.1.3 $\Im(\varphi)$, $\overline{\Im}(\varphi)$ and $\Im^*(\varphi)$

Given that i is the number of elements from I, and k is the number of a predicate's arguments, then i^k is the number of possible k-tuples. Every $\Im(\varphi)$ contains a subset of the i^k possible tuples. Some k-tuples are part of $\Im(\varphi)$, and the rest of the i^k tuples do not occur in $\Im(\varphi)$. In the following, we denote tuples not occurring in $\Im(\varphi)$ by a superscript bar on the tuple, for example, $\overline{(c_2,c_2)}$. Thus, every $\Im(\varphi)$ can be written such that all i^k-tuples are mentioned. Some will have, and the remaining will not have, a superscript bar. We denote interpretations of predicates allowing for tuples with a superscript bar by "$\Im^*(\varphi)$". Thus, instead of

$$\begin{aligned} I = &\quad \{c_1,c_2\}, \\ \Im(F) = &\quad \{(c_1,c_2),(c_2,c_1)\} \end{aligned}$$

one can also write

$$\begin{aligned} I = &\quad \{c_1,c_2\}, \\ \Im^*(F) = &\quad \{(c_1,c_2),(c_2,c_1),\overline{(c_1,c_1)},\overline{(c_2,c_2)}\}. \end{aligned}$$

The subset of tuples of $\Im^*(\varphi)$ with a superscript bar we denote by "$\overline{\Im}(\varphi)$". Thus, in the given example, $\Im(F) = \{(c_1,c_2),(c_2,c_1)\}$ and $\overline{\Im}(F) = \{\overline{(c_1,c_1)},\overline{(c_2,c_2)}\}$.

By $\Im^*$, we refer to interpretations consisting of $\Im^*(\varphi)$, instead of $\Im(\varphi)$.

6.3.2 Classical Semantics vs. New Semantics

In this section, we schematically contrast the conceptions of classical semantics and New Semantics of Q. We hereby take up the analogy of arithmetical experiments and defining $\Im_T$ and $\Im_F$ according to classical semantics, cf. sections 3.1.4. From this, the problem of semantics arises, which shall be solved by New Semantics, cf. section 3.1.6.

In classical semantics, we must examine whether each $\Im$ is a model or a counter-model of a *wff* A. $\Im_T$ and $\Im_F$ cannot be generated without examining single interpretations $\Im$. We represent this by the following schema:

$$\begin{tikzcd} & \Im \models A \Rightarrow \Im_T \\ A, \Im \arrow[ur, dotted] \arrow[dr, dotted] & \\ & \Im \not\models A \Rightarrow \Im_F \end{tikzcd}$$

One consequence of this conception is that $\Im_T$ and $\Im_F$ are not completely determined. If I contains an infinite number of values, the rules defining models cannot decide whether an $\Im$ is a model or a counter-model of the formula in question. As a result, one must rely on intuition, or some standard interpretation that is not algorithmically evaluable, to manage $\Im$ with infinite I. This is relevant to identifying tautologous or contradictory formulae because *wff*s exist that have models or counter-models only in the case of $\Im$ with infinite I (cf. p. 299). That is why we use a spotted arrow in the schema. It expresses that the assignment of an $\Im$ to $\Im_T$ and $\Im_F$ is not completely determined by iteratively applying the rules defining models. Strictly speaking, this definition can only determine, for a finite number of finite $\Im$, whether they are models or counter-models. Therefore, it does not determine the conditions of truth or falsehood of a *wff*s *in total*, just as "arithmetical experiments" do not determine an infinite series of digits.

We do not maintain that classical semantics is mistaken. Rather, any alternative must be measured against classical semantics. This means that any alternative semantics should assign the same models and counter-models to a *wff* A as classical semantics does. However, there is no need to do this in the same way. There may be alternative, more favorable methods of identifying $\Im_T$ and $\Im_F$. New Semantics provides such a method.

New Semantics does not depend on the cumbersome examination of single interpretations. Instead, on the basis of a *wff* A alone, an ab-expression is constructed mechanically in a finite number of steps. This ab-expression identifies the structural features of subclasses of $\Im_T$ and $\Im_F$ such that it is possible to construct $\Im_T$ and $\Im_F$ systematically, without deciding if each single interpretation is a model or counter-model. By "structural features" of subclasses of $\Im_T$ and $\Im_F$, we mean features common to a class of models / counter-models due to which the single interpretations of those classes are models / counter-models. This will be explained in detail in section 6.3.4. In sections 6.3.3 and 6.3.5, we will elaborate on how to interpret ab-expressions as descriptions of structural features of $\Im$, and how to use them as construction rules for $\Im_T$ and $\Im_F$. Considering the construction of ab-expressions via the construction of A_T and A_F, the conception of New Semantics can be expressed schematically as follows:

$$\begin{tikzcd} & A_T \Rightarrow a - PG \Rightarrow \Im_T \\ A \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\ & A_F \Rightarrow b - PG \Rightarrow \Im_F \end{tikzcd}$$

The significant difference between New Semantics and classical semantics is that the differentiation of $\Im_T$ and $\Im_F$ is not based on examining single interpretations. Only based upon some standard expressions (A_T / A_F and $a - PG$ /

$b - PG$) can one construct the class of models and counter-models by their syntactical features. According to a Wittgensteinian view, this difference is crucial. As the notation of continued fractions allows us to "recognize a law in the numbers" in the case of real numbers such as $\sqrt{2}$, π and e, the ab-expressions allow us to "recognize a law" in the class of models and the class of counter-models (cf. p. 100). As a result, one need not examine single interpretations $\Im$ to classify them as models or counter-models. Instead, the structure of models and counter-models is defined by syntactic properties that allow generation of the totality of models and counter-models of a certain *wff* A. Because of this, New Semantics makes it possible to determine the class of models $\Im_T$ and counter-models $\Im_F$ *in total* because it allows one to specify $\Im_T$ and $\Im_F$ by iteration.

6.3.3 Paraphrase

This section explains the interpretation of ab-expressions. When we say, "interpretation of an ab-expression," we mean a translation of ab-expressions into standardized ordinary language expressions whose meanings are known. Such a translation shall be called a "paraphrase".

To understand ab-expressions, the meaning of the symbolizing properties must be understood. Each symbolizing property characterizes a feature of $\Im_T$ or $\Im_F$. By virtue of the previous explanations, and due to the transparency of the ab-expressions, it should be principally clear how to paraphrase ab-expressions. This section precisely defines the rules of their paraphrases.

To define the rules of paraphrasing ab-expressions, additional terminology must be introduced.

combined quantifiers: Quantifiers of a complex pole's prefix are combined iff they are combined according to the following rules:

1. Two quantifiers $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of a pole are combined if $\mathcal{Q}_1$ is succeeded by a number μ, and $\mathcal{Q}_2$ by a number ν, and μ and ν occur in the same line (the same propositional function).
2. If a quantifier $\mathcal{Q}_1$ is combined with a quantifier $\mathcal{Q}_2$ on the left of $\mathcal{Q}_1$, and $\mathcal{Q}_2$ is combined with a quantifier $\mathcal{Q}_3$ on the left of $\mathcal{Q}_2$, then $\mathcal{Q}_1$ and $\mathcal{Q}_3$ are combined.

quantifier-succession: A quantifier-succession consists of all those quantifiers that are combined with each other.

prefix-succession: A prefix-succession consists of the sequence of names in the prefix and a quantifier-succession of the prefix.

EXAMPLE: In the following pole

$$\begin{array}{llllll} & b_4,\ \exists_2 & & & & {}^{3}\ \exists_1\ -a{-}F_{1234} \\ & & & \exists & \langle & \\ a_4, & & \forall_1 & & & {}_{3}\ \forall_2\ -a{-}G_{1234} \end{array}$$

the two quantifiers $\exists_1$ and $\forall_2$ of the inmost column of quantifiers, and only these two quantifiers, are not combined with each other. Thus, the pole has the following two quantifier-successions:

$${}^{\exists_2}{}_{\forall_1}\exists{<}^{3}_{3},{}^{\exists_1} \text{ and}$$

$${}^{\exists_2}{}_{\forall_1}\exists{<}^{3}_{3}{}_{\forall_2}.$$

And the following two prefix-successions:

$${}_{a_4,}{}^{b_4,\ \exists_2}{}_{\forall_1}\exists{<}^{3}_{3},{}^{\exists_1} \text{ and}$$

$${}_{a_4,}{}^{b_4,\ \exists_2}{}_{\forall_1}\exists{<}^{3}_{3}{}_{\forall_2}.$$

The distinction of combined and non-combined quantifiers is relevant for identifying $\Im_T$ and $\Im_F$. In closed structures, this difference is represented by the scope of the quantifiers. Thus, one could equivalently identify combined quantifiers by referring to the scope of quantifiers in closed structures:

combined quantifiers: Two quantifiers $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of a pole $\mathcal{P}$ are combined iff the translation of $\mathcal{Q}_1$ occurs in the closed structure, which is the translation of $\mathcal{P}$, in the scope of the translation of $\mathcal{Q}_2$.

Regarding the identification of $\Im_T$ and $\Im_F$, the relevant difference of combined and non-combined quantifiers concerns (i) the order of existential and universal quantifiers, and (ii) the relation of universal quantifiers. The order of existential and universal quantifiers is only a symbolizing property within a quantifier-sequence. This shall be demonstrated by the following examples:

no.	**closed structure**	**pole**
(1)	$\exists x \forall y \exists z (Fxz \wedge Gyz)$	${}^{\exists_1}{}_{\forall_1}\exists < \begin{matrix} 2 & -a-F_{12} \\ 2 & -a-G_{12} \end{matrix}$
(2)	$\forall y \exists z (\exists x Fxz \wedge Gyz)$	${}_{\forall_1}\exists < \begin{matrix} 2\,{,}\exists_1 & -a-F_{12} \\ 2 & -a-G_{12} \end{matrix}$
(3)	$\exists z (\exists x Fxz \wedge \forall y Gyz)$	$\exists < \begin{matrix} 2\ \exists_1 & -a-F_{12} \\ 2\ \forall_1 & -a-G_{12} \end{matrix}$

(1) is only true if some object at the 1. position of the 2-tuples of $\Im(F)$, combined with all objects at the 1. position of the 2-tuples of $\Im(G)$, combined with some object, the same at the 2. position of the 2-tuples of $\Im(F)$ and the 2. position of the 2-tuples of $\Im(G)$, satisfies $\Im^*(F)$ and $\Im^*(G)$. In contrast, (2) is already true if all objects at the 1. position of the 2. tuples of $\Im(G)$, combined with some object, the same at the 2. position of the 2-tuples of $\Im(F)$ and the 2. position of the 2-tuples of $\Im(G)$, combined with some object at the 1. position of the 2-tuples of $\Im(F)$, satisfy $\Im^*(F)$ and $\Im^*(G)$. That is to say, the following $\Im$ is a model of (2), but a counter-model of (1):

$I = \{c_1, c_2\}$,

$\Im(F) = \{(c_1, c_1), (c_2, c_2)\}$,

$\Im(G) = \{(c_1, c_1), (c_2, c_2)\}$.

In contrast, the following $\Im$ is also a model of (1):

$I = \{c_1, c_2\}$,

$\Im(F) = \{(c_1, c_1), (c_1, c_2)\}$,

$\Im(G) = \{(c_1, c_1), (c_2, c_2)\}$.

Thus, the order of $\exists_1$ and $\forall_1$ makes a difference in the truth conditions of (1) and (2). In contrast, $\exists_1$ and $\forall_1$ are not combined in (3). Contrary to (1) and (2), the relative position of these two quantifiers to each other is not a symbolizing property in (3). Instead, the two quantifier-successions of (3) identify separate conditions of the models of (3). Thus, the paraphrases of the non-combined quantifiers are not connected by "combined with," but rather by "and". (3) is true iff some object, the same at the 2. position of the 2-tuples of $\Im(F)$ and the 2. position of the 2-tuples of $\Im(G)$, combined with some object at the 1. position of the 2-tuples of $\Im(F)$, *and* all objects at the 1. position of the 2-tuples of $\Im(G)$, satisfies $\Im^*(F)$ and $\Im^*(G)$. This condition is satisfied, for example, by the following $\Im$:

$I = \{c_1,c_2\}$,

$\Im(F) = \{(c_1,c_1)\}$,

$\Im(G) = \{(c_1,c_1),(c_2,c_1)\}$.

Furthermore, quantifier-successions with n universal quantifiers refer to m^n combinations of objects, where m is the number of objects in the domain I. On the other hand, universal quantifiers from different quantifier-successions refer to the objects of the domains separately, without combining them. This becomes clear with the following examples (cf. p. 280f.):

no.	**closed structure**	**pole**
(4)	$\forall x \forall y \exists z(Fxz \land Gyz)$	${}^{\forall_1}{}_{\forall_1}\exists < \substack{2\ -a-F_{12} \\ 2\ -a-G_{12}}$
(5)	$\exists z(\forall x Fxz \land \forall y Gyz)$	$\exists < \substack{2\ \forall_1\ -a-F_{12} \\ 2\ \forall_1\ -a-G_{12}}$

The following $\Im$, for example, is not a model of (4):

$I = \{c_1,c_2\}$,

$\Im(F) = \{(c_1,c_1),(c_2,c_2)\}$,

$\Im(G) = \{(c_1,c_1),(c_2,c_2)\}$.

The following $\Im$ is a model of (4), as all objects at the 1. position of the 2-tuples of $\Im(F)$, *combined with* all objects at the 1. position of the 2-tuples of $\Im(G)$, combined with the same object at the 2. position of the 2-tuples of $\Im(F)$ and $\Im(G)$, satisfy $\Im^*F$ and $\Im^*G$:

$I = \{c_1,c_2\}$,

$\Im(F) = \{(c_1,c_1),(c_2,c_2)\}$,

$\Im(G) = \{(c_1,c_1),(c_2,c_1),(c_1,c_2),(c_2,c_2)\}$.

Here, all 2^2 combinations of objects at the 1. position of the 2-tuples of $\Im(F)$, and the 1. position of the 2-tuples of $\Im(G)$, are combined with the same object at the 2. position of the 2-tuples of $\Im(F)$ and $\Im(G)$.

In contrast, (5) demands that an object that is the same at the 2. position of the 2-tuples of $\Im(F)$, and at the 2. position of the 2-tuples of $\Im(G)$, be combined with all objects at the 1. position of 2-tuples of $\Im(F)$, *and* with all objects at the

1. position of the 2-tuples of $\Im(G)$. There is no need to refer to m^n combinations of objects. Instead, the same object at the 2. positions of $\Im(F)$ and $\Im(G)$ must be combined with all m objects at the 1. position of the 2-tuples of $\Im(F)$ and with all m objects at the 1. position of the 2-tuples of $\Im(G)$. Thus, the following $\Im$ is a model of (5):

$I = \{c_1,c_2\}$,

$\Im(F) = \{(c_1,c_1),(c_2,c_1)\}$,

$\Im(G) = \{(c_1,c_1),(c_2,c_1)\}$.

By referring to combined quantifiers, quantifier-successions, and prefix-successions, the general rules of paraphrasing ab-expressions are defined as follows.

rules of paraphrase:

1. The paraphrase of the a-pole-groups of a *wff* A begins with "$\Im$ is a model of A iff," and the paraphrase of the b-pole-groups begins with "$\Im$ is a counter-model of A iff".
2. The paraphrases of the a-pole-groups / b-pole-groups are connected by "or".
3. The paraphrases of the poles are connected by "and".
4. Propositional poles of form $a - \mathcal{J}$ are paraphrased by "$\Im(\mathcal{J}) = T$", propositional poles of form $a - \mathcal{J}$ are paraphrased by "$\Im(\mathcal{J}) = F$".
5. Predicate poles are paraphrased as follows:
   (a) The existential quantifier ($\exists$) is paraphrased by "some object", while the universal quantifier ($\forall$) is paraphrased by "all objects", names (t) by "$\Im(t)$".
   (b) Closed forks are paraphrased by "the same", and open forks by "distributed among". If closed and open forks are nested, case differentiations are introduced. In each case, one has to proceed by paraphrasing the subsequent forks.
   (c) A number k subsequent to a fork in the prefix is paraphrased by "at the k. position of the n-tuples of $\Im(\varphi)$" if $a - \varphi_{1\ldots n}$ occurs in the same line as k, and by "at the k. position of the n-tuples of $\overline{\Im}(\varphi)$" if $b - \varphi_{1\ldots n}$ occurs in the same line as k. The paraphrases of the numbers succeeding the forks are connected by "and".

(d) Poles are paraphrased from left to right. The elements of a prefix-succession are connected by "combined with". If quantifiers are not combined, case distinctions are introduced: in each case, one proceeds by paraphrasing the quantifiers. The different cases of non-combined quantifiers are connected by "and".

(e) The paraphrase ends by paraphrasing the propositional functions by "$\Im^*(\varphi)$". The single paraphrases of propositional functions are connected by "and". The paraphrases of the propositional functions follow "satisfies," if the prefix begins with a name or the existential quantifier, or "satisfy," if the prefix begins with a universal quantifier.

EXAMPLE 1: Let A be the following *wff*:

$$\exists y \forall x(\neg Fx \wedge Gy \vee Hxy).$$

A_T:

$$\begin{array}{l} \exists y \forall x Hxy \vee \\ \exists y(\forall x(\neg Fx \vee Hxy) \wedge Gy). \end{array}$$

Translation into a-pole-groups:

$$\begin{array}{l} a-\{\exists_2\forall_1 - a - H_{12}\}, \\ a-\left\{\begin{array}{lll} & & \forall < \begin{array}{l} {}^{1}\ -b-F_1 \\ {}_{1}\ -a-H_{12} \end{array} \\ \exists < \begin{array}{l} {}^{2} \\ {}_{1} \end{array} & & \qquad -a-G_1 \end{array}\right\}. \end{array}$$

For the sake of clarity, we provide a structure of the paraphrase of the a-pole-groups:

- $\Im$ is a model of $\exists y \forall x(\neg Fx \wedge Gy \vee Hxy)$ iff
  -
    - some object at the 2. position of the 2-tuples of $\Im(H)$, combined with
    - all objects at the 1. position of the 2-tuples of $\Im(H)$,
    - satisfies $\Im^*(H)$;

  or

- - some object, the same at the 2. position of the 2-tuples of $\Im(H)$ and the 1. position of the 1-tuples of $\Im(G)$, combined with
    - all objects, distributed among the 1. position of the 1-tuples of $\overline{\Im}(F)$ and the 1. position of the 2-tuples of $\Im(H)$,
    - satisfies $\Im^*(F)$ and $\Im^*(G)$ and $\Im^*(H)$.

EXAMPLE 2: Let A be the following A_T:

$$\exists y_3(\exists y_1 \forall x_1 F y_1 x_1 y_3 \wedge \exists y_2 \forall x_2 G y_2 x_2 y_3).$$

Translation into a-pole-groups:

$$a - \left\{ \exists < \begin{array}{llll} ^3 & \exists_1 & \forall_2 & -a-F_{123} \\ _3 & \exists_2 & \forall_2 & -a-G_{123} \end{array} \right\}.$$

Paraphrase (case distinctions are numerated):

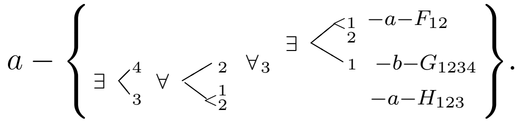
- $\Im$ is a model of $\exists y_3(\exists y_1 \forall x_1 F y_1 x_1 y_3 \wedge \exists y_2 \forall x_2 G y_2 x_2 y_3)$ iff
  - some object, the same at the 3. position of the 3-tuples of $\Im(F)$ and at the 3. position of the 3-tuples of $\Im(G)$, combined with
    1. - some object at the 1. position of the 3-tuples of $\Im(F)$, combined with
       - all objects at the 2. position of the 3-tuples of $\Im(F)$,

       and
    2. - some object at the 1. position of the 3-tuples of $\Im(G)$, combined with
       - all objects at the 2. position of the 3-tuples of $\Im(G)$,
  - satisfies $\Im^*(F)$ and $\Im^*(G)$.

EXAMPLE 3: Let A be the following A_T:

$$\exists y_1 \forall x_2(\forall x_1 \exists y_2(F y_2 y_2 \wedge \neg G y_2 x_2 x_1 y_1) \vee H x_2 x_2 y_1).$$

Translation into a-pole-groups:



Paraphrase:

- $\Im$ is a model of $\exists y_1 \forall x_2 (\forall x_1 \exists y_2 (Fy_2y_2 \wedge \neg Gy_2x_2x_1y_1) \vee Hx_2x_2y_1)$ iff
    - some object, the same at the 4. position of the 4-tuples of $\overline{\Im}(G)$ and the 3. position of the 3-tuples of $\Im(H)$, combined with
    - all objects distributed among
        1. the 2. position of the 4-tuples of $\overline{\Im}(G)$, and
        2. the same at the 1. position of the 3-tuples of $\Im(H)$, and the 2. position of the 3-tuples of $\Im(H)$,

      combined with
    - all objects at the 3. position of $\overline{\Im}(G)$, combined with
    - some object, the same at the 1. position of the 2-tuples of $\Im(F)$ and at the 2. position of the 2-tuples of $\Im(F)$, and at the 1. position of the 4-tuples of $\overline{\Im}(G)$,
    - satisfies $\Im^*(F)$ and $\Im^*(G)$ and $\Im^*(H)$.

Due to the intricacies of ordinary language, and the limited perceptivity of human mind, the paraphrases become incomprehensible in case of increasing complexity. However, the main purpose of the paraphrases is not to make single ab-expressions, or *wff*s from which the ab-expressions result, understandable. Instead, the purpose is to make understandable the meaning of the syntactical properties of the ab-expressions. This is elucidated by explaining how the ab-expressions determine the truth conditions of *wff*s. If one has understood this, one understands the expressive power of ab-expressions and one will come to understand these condensed expressions better than their paraphrases. We do not know a better way to make truth conditions of *wff*s understandable as by their ab-expressions, or, more precisely, by their ab-symbols. However, it is not the first object of ab-notation to satisfy the *psychological criterion* of developing a comprehensible notation. Instead, the aim is to create a notation satisfying the *logical criterion* of identifying $\Im_T$ and $\Im_F$ by identifying structural features of subclasses of $\Im_T$ and $\Im_F$ by the syntactical features of the notation. This shall be illuminated in the following section.

6.3.4 Structural Features

In this section, we will first explain what it means to say ab-expressions identify structural features of subclasses of $\Im_T$ and $\Im_F$. Then, ab-expressions will be compared with *wff*s, $\bigvee\bigwedge cs$, and ab-symbols. We compare them in light of the demand to identify structural features of subclasses of $\Im_T$ and $\Im_F$.

The paraphrases of ab-expressions make it clear that the poles of a-pole-groups describe features of $\Im^*$ that are common to the elements of a subclass of $\Im_T$ and, by virtue of which, they are models. The single interpretations of this subclass have a common structure, and they differ only in features that are not essential for identifying them as models. We use the term "structural features" only for those features of interpretations satisfied by all interpretations of a subclass of $\Im_T$ / $\Im_F$. Nonessential features of single $\Im^*$ vary within a subclass, whereas the structural features remain constant. All the elements of a class of $\Im$ with the same structure are internally related by their constant structure, while all nonessential features vary systematically. This, we demonstrate with EXAMPLE 1, p. 267. The *wff*

$$\exists y \forall x (\neg Fx \wedge Gy \vee Hxy) \tag{6.22}$$

has the following two a-pole-groups:

$$a - \{\exists_2 \forall_1 \quad - a - H_{12}\},$$

$$a - \left\{ \begin{array}{ccccc} & & \forall & \diagup^{1} & -b - F_1 \\ & \diagup^{2} & & \diagdown_{1} & -a - H_{12} \\ \exists & & & & \\ & \diagdown_{1} & & & -a - G_1 \end{array} \right\}$$

The first a-pole-group describes a class of models that has the following common (structural) features:

- some object at the 2. position of the 2-tuples of $\Im(H)$, combined with
- all objects at the 1. position of the 2-tuples of $\Im(H)$,
- satisfies $\Im^*(H)$.

All other items may vary: the number of objects in I; the interpretations of all propositional variables, all names, and all further predicates; which object at the 2. position of the 2-tuples of $\Im(H)$ is combined with all objects at the 1. position of the 2-tuples of $\Im(H)$; how many further tuples satisfy $\Im(H)$, in addition to the tuples satisfying the aforementioned condition. All these variations are inessential features of single $\Im^*$, and they do not constitute the structure of the $\Im^*$ of the subclass identified by the first a-pole-group.

The second a-pole-group describes a further subclass of $\Im_T$ with the following common features:

- some object, the same at the 2. position of the 2-tuples of $\Im(H)$ and at the 1. position of the 1-tuples of $\Im(G)$, combined with
- all objects, distributed among the 1. position of the 1-tuples of $\overline{\Im}(F)$ and the 1. position of the 2-tuples of $\Im(H)$,
- satisfies $\Im^*(F)$ and $\Im^*(G)$ and $\Im^*(H)$.

Contrary to the subclass with the common structure identified by the first a-pole-group, the subclass identified by the second a-pole-group is characterized by a structure common to $\Im^*(F), \Im^*(G)$, and $\Im^*(H)$. However, some $\Im^*$ may be elements of both subclasses.

The concept of structural features remains imprecise as long as one does not refer to the *identification* of structural features. According to Wittgenstein's terminology, structural features are not "material properties" identified by propositional functions (cf. TLP 4.126[5]). Instead, they are "formal properties," identified by syntactical features of a proper notation. Structural features of classes of interpretations are identified by *syntactical features* of ab-expressions. Thus, there is no way of identifying a certain formal structure except by revealing it using the syntax of proper notation. Syntactical features of expressions in general are the shape (type) of the signs, and their spatial or, in the case of oral expressions, their temporal, composition. The syntactical features of ab-expressions are the shape (type) of their signs as well as the linear order of the signs of the single poles. The vertical order in ab-expressions, and the linear order of poles are not syntactical features that identify structural features. That is to say, according to Wittgenstein's terminology, these syntactical features are not "symbolizing properties" (cf. section 4.2). The syntactical features of ab-expressions serve as *criteria* for identifying structural features of a class of $\Im^*$. By doing so, they make it possible to identify the totality of $\Im_T$ and $\Im_F$. Otherwise, one can only enumerate a finite number of models and counter-models according to classical semantics, just as one is only able to develop finite sequences of the decimal expansion of an irrational number. Such is the case as long as one does not define it by some operation becoming manifest in the structure of a proper notation. There is no way to identify the $\Im_T$ and $\Im_F$ of a *wff* without referring to a proper syntax, because the totality of $\Im_T$ and $\Im_F$ can only be identified by specifying their structure. This is a formal property that is only identifiable by a proper syntax.

Ab-expressions can serve as criteria by virtue of the correspondence of their syntactical features to structural features of subclasses of $\Im^*$. This, we will demonstrate by contrasting the syntax of *wff* (6.22) with the syntax of its a-pole-group.

The *wff*, $\exists y \forall x(\neg Fx \wedge Gy \vee Hxy)$, i.e. (6.22), does not identify structural features of models. This is because its syntactical features cannot be interpreted uniformly such that they identify features of interpretations by virtue of which they are models. For example, the single signs of (6.22) and their spatial composition do not determine under what conditions the same object must satisfy the 1. position of the 1-tuples of $\Im(G)$ and the 2. position of the 2-tuples of $\Im(H)$ combined with objects at the 1. position of the 2-tuples of $\Im(H)$ in order for $\Im^*$ to be a model; nor do they determine under what conditions this need not be the case. However, these must be determined to identify structural features of models because not all models satisfy the condition that the same object occurs at the 1. position of the 1-tuples of $\Im(G)$ and the 2. position of the 2-tuples of $\Im(H)$. Meanwhile, for some models, this feature must be satisfied in order for them to be models. No constant feature in the class of all models corresponds to the syntactical feature that the existential variable y occurs at several argument positions in (6.22). Thus, this syntactical feature cannot be interpreted uniformly. The same is valid for the universal variable x: x occurs to the left and right of $\vee$ in (6.22) and cannot be interpreted uniformly such that all objects merely are distributed among the 1. position of the 1-tuples of $\overline{\Im}(F)$ and the 1. position of the 1-tuples of $\Im(H)$. This is only true for a certain subclass of the models; for another subclass, all objects must occur at the 1. position of the 1-tuples of $\Im(H)$. The sentential connectives cannot be interpreted uniformly, either. $\wedge$ neither connects single descriptions of structural features of a class of models, nor does it correspond to some constant feature in the class of models. The same holds for $\vee$. It neither connects descriptions of different classes of models, nor does it correspond to some constant feature of a certain class of models. Therefore, the syntactical features of $\exists y \forall x(\neg Fx \wedge Gy \vee Hxy)$ do not identify structural features of subclasses of $\Im_T$. The same holds for the paraphrases of *wff*s.[12] "Some y satisfies all x satisfy: .not Fx and Gy. or Hxy:" does not identify the truth conditions of *wff*s, because this paraphrase does not paraphrase syntactical features corresponding to constant structural features of subclasses of $\Im_T$ or $\Im_F$. Paraphrases of *wff*s suffer from the same deficiencies as the syntax of *wff*s. For this reason, paraphrases of *wff* do not clarify their truth conditions and do not contribute to the semantics of predicate logic.

In contrast, paraphrases of ab-expressions do clarify the truth conditions of *wff*s because, by relying on the syntactical features of ab-expression, they ex-

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[12] For a definition of a mechanical procedure to paraphrase *wff*s by a standardized language cf. Lampert (2005a), chapter 9.1 or Kalish (1964).

press the structural features of $\Im_T$ and $\Im_F$. The a-pole-groups identify different subclasses of models. The syntactical features of the single poles correspond to features common to a subclass of interpretations, by virtue of which they are models; every structural difference is represented uniformly by some syntactical difference. Only in the second a-pole-group does the existential quantifier refer to argument positions of different propositional functions. Only the models identified by this pole-group are models in virtue of the fact that the same object occupies the 2. position of the 2-tuples of $\Im(H)$ and the 1. position of the 1-tuples of $\Im(G)$. This structural feature is identified by the syntactical feature of a closed fork connecting the numbers of the respective argument positions. In contrast, in the first a-pole-group, the existential quantifier refers only to the 2. position of $\Im(H)$, because this pole-group identifies a sufficient condition of models that does not involve any constraints to $\widehat{\Im(\varphi)}$ besides $\Im(H)$. Analogously, the connection of numbers by an open fork that follows the universal quantifier in the second a-pole-group is a syntactical feature corresponding to the structural feature that all objects from I are distributed among the 1. position of the 1-tuples of $\overline{\Im}(F)$ and the 1. position of the 2-tuples of $\Im(H)$. However, in the first a-pole-group, no fork succeeds the universal quantifier. In the class of models identified by this pole-group, all objects must satisfy the 1. position of the 2-tuples of $\Im(H)$.

As the rules of paraphrasing an ab-expression make clear by the position of quantifiers, the connection of numbers by forks, the type of quantifiers and names, the innermost poles preceding predicates, and the occurrences of poles in a- and b-pole-groups, it is determined, in general, how the objects from I must be distributed to the respective positions of tuples from $\Im^*(\varphi)$ to satisfy conditions of subclasses of $\Im_T$ and $\Im_F$. In contrast, the syntax of arbitrary *wff*s cannot be interpreted in this way because its features do not correspond uniformly to structural features of $\Im_T$ and $\Im_F$. For example, occurrences of the same variable at different positions, the order of quantifiers, the shape of the sentential connectives or the shape of the variables cannot be given a unique meaning in terms of identifying features of $\Im^*$, which serve as criteria for models or counter-models of the *wff* in question.

In contrast to arbitrary *wff*s, the $\bigvee\bigwedge cs$ A_T and A_F do identify structural features. Their syntax corresponds to the syntax of ab-expressions, and therefore also to structural features of $\Im^*$. However, there are several differences between the syntax of $\bigvee\bigwedge cs$ and ab-expressions that cause the syntax of ab-expressions to be preferable to the syntax of $\bigvee\bigwedge cs$ for identifying structural features of $\Im^*$.

1. ab-expressions are enumerations of classes. By their syntax, it is clear that the order of the pole-groups, as well as the order of poles of a pole-group, are not symbolizing properties. In other words, enumerations have no syntactical features that might identify a structural feature. In contrast, a *wff* expressed in terms of disjunctions and conjunctions has a structure. It does not follow from the mere use of $\wedge$ and $\vee$ that the order of conjuncts and disjuncts does not symbolize in the special case of $\bigvee \bigwedge cs$. Instead, this has to be justified by equivalence rules (ASS$\wedge$, COM$\wedge$, ASS$\vee$, COM$\vee$). Similarly, by enumerations of existential / universal quantifiers in a sequence of existential / universal quantifiers, it is expressed that the special order of the quantifiers does not symbolize, whereas in the case of $\bigvee \bigwedge cs$, this has to be justified by equivalence rules.

2. Poles are two dimensional, whereas closed structures are linear strings. By their two-dimensionality, symbolizing features of poles resulting from the ordering of signs can be identified generally, without referring to the special signs that are ordered. The linear ordering of a pole's signs is significant, while their vertical ordering is insignificant. In contrast, in closed structures, one must distinguish significant and insignificant linear orderings based on the ordered signs and their occurrences within the structure of the formula. The order of disjuncts within the disjunction and the order of conjuncts within the conjunctions are insignificant, while the order of quantifiers is significant, at least insofar as an existential quantifier occurs in the scope of an universal quantifier or vice versa.

3. Contrary to ab-expressions, some differences concerning the structural features of $\Im_T/\Im_F$ are not symbolized by different kinds of syntactical features of $\bigvee \bigwedge cs$. $\vee$ and $\wedge$ occur in A_T within the scope of quantifiers in closed structures, as well as outside of closed structures. In both contexts, however, they contribute differently to the identification of structural features, which becomes clear with the different corresponding syntactical features in ab-expressions (commas vs. forks). Outside the scope of quantifiers, $\wedge$ and $\vee$ signify whether structural features, of the same ($\wedge$) or of different subclasses of $\Im_T$ / $\Im_F$ ($\vee$), are identified by closed structures. In contrast, within the scope of quantifiers, they determine the kind of structural features a subclass of $\Im_T$ / $\Im_F$ has to satisfy. Similarly, in $\bigvee \bigwedge cs$, variables and names are symbolized with small letters. But unlike in the case of names, the specific type of the variables is not a symbolizing property. Instead, what sym-

bolizes are the locations of the bound variables. This structural difference is taken into account in the syntax of the ab-notation by replacing variables with numbers.

4. On the other hand, $\bigvee\bigwedge cs$ also contain syntactical differences, while there is a structural similarity in the $\Im$*. Thus, as an example, the existential quantifier may occur in different contexts within a closed structure, namely succeeding a propositional function (e.g. $\exists x Fxx$), a conjunction (e.g. $\exists x(Fx \wedge Gx)$), or an universal quantifier (e.g. $\exists x \forall y(Fxy \vee Gxy)$). In each case, it remains unclear whether the relationship between the existential quantifier and the respective argument positions of the bound variable is the same. On the contrary, by virtue of the use of closed forks in the syntax of ab-expressions, it becomes clear that the relation of the existential quantifier to the respective argument positions is always the same. In each case, the quantifier determines that the *same* object must satisfy the respective positions of certain tuples of the concerned $\Im(\varphi)$.

5. The syntactical indeterminacy of the relationship between quantifiers and argument positions in closed structures becomes even more evident in the case of the universal quantifier. Whereas in different contexts, such as $\forall x Fxx$ and $\forall x \exists y(Fxy \wedge Gxy)$, the relation is the same, in other contexts, such as $\forall x(Fx \vee Gx)$, the relation is different. This difference is identified by the syntactical difference between closed and open forks in the ab-expression. The deficiency of the syntax of $\bigvee\bigwedge cs$ that invokes these indeterminacies lies in the fact that it does not determine, by syntax, what symbolizes the kind of relationship that exists between a quantifier and the argument positions. This becomes evident by comparing the following closed structures with their representation as complex poles:

   $$\begin{array}{ll} \forall x \exists y(Fxy \wedge Gxy) & \forall < {}^{1}_{1}\exists < \begin{smallmatrix} 2 \ -a-F_{12} \\ 2 \ -a-G_{12} \end{smallmatrix} \\ \forall x \exists y \forall z(Fxyz \vee Gxyz) & \forall < {}^{1}_{1}\exists < {}^{2}_{2}\forall < \begin{smallmatrix} 3 \ -a-F_{123} \\ 3 \ -a-G_{123} \end{smallmatrix} \end{array}$$

   From the syntax of predicate logic, it remains undetermined whether the relation between the respective outermost universal quantifier and its argument positions is the same in both formulae or not. In both closed structures, the outermost universal quantifier precedes an existential quantifier. In this sense, one has a syntactical similarity. On the other hand, in the former closed structure, the variable that is bound by the outermost universal quantifier occurs in different conjuncts, while in the latter closed structure, the variable bound by the outermost universal quantifier occurs in different disjuncts. By the syntax of predicate logic, it cannot be determined what the symbolizing syntactical feature is in this case. Only by the syntax of the ab-notation does it become clear, through the use of closed and open forks, that there is a significant difference in the relation of the universal quantifier to the respective argument positions (cf. also p. 232 and p. 234). The use of closed and open forks elucidates that it is the occurrence of the bound variable, not the position of the universal quantifier, that is pivotal for the universal quantifier's relation to argument positions in closed structures.

6. There is another difference between the use of forks in the ab-notation and the use of sentential connectives in the predicate syntax. According to the classical interpretation of *wff*s, one interprets a formula moving from the inside to the outside according to its logical hierarchy (cf. the definition of models on p. 259). However, by doing this, it is impossible to systematically construct models or counter-models. Given such a simple formula as $\forall x(Fx \vee Gx)$, no information is available starting from the inside that tells one how many objects must satisfy the argument positions of the involved propositional functions. This does not become clear until the universal quantifier is considered. For this reason, classical semantics do not construct models and counter-models systematically. They merely examine whether given interpretations (with finite I) are models or counter-models. In contrast, poles are paraphrased from the outside to the inside, and the paraphrases of their prefixes successively specifies the distribution of objects to the argument positions of the propositional functions. This is due to the use of forks. In closed structures, one cannot proceed similarly because the quantifiers are not succeeded by forks, but simply by the bound variable. One must refer to the occurrences of the variables in the inside of the closed structures to (i) identify the occurrences of the respective variable in propositional functions and to (ii) identify the logical connections of these propositional functions. Thus, if one wants to paraphrase closed structures proceeding from the outside to the inside, one must actually translate them to their corresponding poles. Taking into account the aforementioned deficiencies (1)-(5), one can generally say that a correct paraphrase of closed structures in terms of identity criteria of models and counter-models

amounts to a paraphrase of their corresponding pole in the ab-notation. This demonstrates the merit of the syntax of ab-expressions in comparison with the predicate notation of $\bigvee\bigwedge cs$.

The predicate syntax is deficient for identifying structural features of $\Im_T$ and $\Im_F$, even in case of $\bigvee\bigwedge cs$. This is because $\vee$ and $\wedge$ are interpreted differently inside and outside the scope of quantifiers regarding the identification of structural features of $\Im^*$ (cf. 3.), because the specific relation of quantifiers to argument positions is not expressed explicitly (cf. 4.-6.), and also because it is not determined by syntactical constraints which syntactical features are symbolizing properties (1.-6.).

Regarding the interpretations of logical constants, i.e., quantifiers and sentential connectives in *wff*s, three kinds of interpretations must be distinguished:

1. their interpretation according to classical semantics (cf. *rules 6* and *7* on p. 259).
2. their interpretation in terms of identifying structural features of $\Im^*$. This interpretation is only applicable to their occurrence in A_T and A_F, and it is actually carried out by the paraphrases of the corresponding ab-expressions.
3. their interpretation by means of rules that translate arbitrary *wff*s into expressions of a syntax interpretable as identity criteria of $\Im_T$ and $\Im_F$. In this sense, logical constants are interpreted by equivalence rules. The quantifiers,for example, are interpreted by applying equivalence rules specified by $\exists$-R. and $\forall$-R.

As $\bigvee\bigwedge cs$ are themselves *wff*s, they can be interpreted in terms of all three kinds of interpretations. In contrast, ab-expressions are only interpretable in the sense of identity criteria of $\Im_T$ and $\Im_F$ according to their paraphrase. This uniqueness of interpretation is a further argument for using ab-expressions as identity criteria of $\Im_T$ and $\Im_F$.

Unlike the syntax of *arbitrary wff*s, the syntactical features of ab-expressions, as well as A_T and A_F, can be interpreted as identity criteria of $\Im_T$ and $\Im_F$. However, this characteristic of the syntax of ab-expressions and $\bigvee\bigwedge cs$ only satisfies a necessary, but not a sufficient, condition of an ideal logical notation.[13] An ideal

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[13] According to the concept of an ideal logical notation and its tradition cf. Brun (2004, 2nd edition), chapter 7 and Evans (1985), p. 71. By the concept of an ideal logical notation elaborated here, we claim to specify Wittgenstein's idea of it.

notation of predicate logic, in terms of *New Logic*, has to represent the truth conditions of *wff*s unambiguously. That is to say, not only do the differences of the $\Im_T$ and $\Im_F$ have to correspond to syntactical differences of an ideal expression, but also the differences in the syntax of an ideal expression must correspond (uniformly) to differences of the $\Im_T$ and $\Im_F$. The syntactical differences of the ideal expressions must have the same *manifold* as the structural differences of the $\Im_T$ and $\Im_F$. This requirement is not satisfied by ab-expressions in general. For example, the two a-pole-groups (6.23) and (6.24) differ:

$$a - \{\forall_1 - a - F_1, \exists_1 - a - F_1\} \tag{6.23}$$

$$a - \{\forall_1 - a - F_1\} \tag{6.24}$$

However, no structural difference in the class of models corresponds to the syntactical difference of (6.23) and (6.24), because the two respective A_T, namely $\forall xFx \wedge \exists xFx$ and $\forall xFx$, are equivalent. Similarly, the two lists of a-pole-groups (6.25) and (6.26) differ without identifying different models:

$$a - \{\forall_1 - a - F_1\},\ a - \{\exists_1 - a - F_1\} \tag{6.25}$$

$$a - \{\exists_1 - a - F_1\} \tag{6.26}$$

The a-pole-groups of (6.25) identify two subclasses of models. However, the first subclass is a subclass of the second subclass. Therefore, it does not specify the identification of models. Syntactical features may also occur that identify inconsistent structural features as in the following case:

$$a - \{\forall_1 - a - F_1, \exists_1 - b - F_1\} \tag{6.27}$$

This a-pole-group identifies an empty class of models. However, the non-existence of a-pole-groups also identifies an empty class of models. Thus, again, there is a syntactical difference that does not correspond to a structural difference. In all these cases, we have redundant syntactical features which do not specify the structure of models / counter-models. The structural features that are identified can be ignored without invoking a variation of the class of $\Im_T$ and $\Im_F$. This also applies to complex poles and closed structures, respectively, as the following examples show. The *cs* $\exists x(\forall yFxy \wedge \exists yFxy)$ is equivalent to the c.s $\exists x\forall yFxy$; the *cs* $\forall x(\exists yFxy \vee \forall yFxy)$ is equivalent to the *cs* $\forall x\exists yFxy$; and the *cs* $\exists x(Fx \wedge$

$\neg Fx)$ does not identify any consistent structure of $\Im$. Thus, the general concepts of a complex pole and that of a closed structure do not identify *ideal unanalysable expressions* which contain no insignificant syntactical features.

The deficiency of containing redundant syntactical features is overcome by referring to *ab-symbols* or *reduced* A_T and A_F. These unambiguously represent the conditions of truth and falsehood of *wff*s. The syntax of $\bigvee\bigwedge cs$ and ab-expressions is already restricted compared to the syntax of arbitrary *wff*s. As a result, many equivalent, though syntactically different, *wff*s are represented by the same $\bigvee\bigwedge cs$ / ab-expression. However, their syntax is not restricted to the extent that all equivalent *wff*s are represented by the same $\bigvee\bigwedge cs$ / the same ab-expression. In this respect, the procedure resulting in $\bigvee\bigwedge cs$ is similar to the usual normal form procedures, such as CDNF, in propositional logic. Only by supplementing the transformation of *wff*s to $\bigvee\bigwedge cs$ by a *procedure of minimizing* $\bigvee\bigwedge cs$ to *reduced* $\bigvee\bigwedge cs$ does one derive unambiguous representatives of the conditions of truth and falsehood of *wff*s. An ideal logical notation, in terms of Wittgenstein's *New Logic*, presumes a solution of the equivalence problem. Thus, the conception of New Semantics is not realized to its full extent until a procedure of minimizing $\bigvee\bigwedge cs$ / ab-expressions is defined, such that all equivalent formulae are represented by the same reduced $\bigvee\bigwedge cs$ / the same ab-symbol. ab-symbols are the only expressions of an *ideal* logical notation because their syntactical structure corresponds one-to-one to the structure of the $\Im_T$ and $\Im_F$. By means of this correspondence, they serve as ideal identity criteria of the models and counter-models of *wff*s.

The relations of *wff*s, ab-expressions and ab-symbols can be summarized by the following table:

	wff	ab-expressions	ab-symbols
difference	do not identify structural features	identify structural features	identify structural features unambiguously
Q-expression	*wff*	$\bigvee\bigwedge gS$	reduced $\bigvee\bigwedge gS$

According to New Semantics, a proper semantics of predicate logic does not consist of a correlation of *wff*s to objects, facts, states of affairs, possible worlds, or other such extra-linguistic entities. Instead, a proper semantics is based upon a translation of *wff*s, in the expression of an ideal notation identifying the truth conditions of the *wff*s, unambiguously. This conception of semantics is realized by solving the equivalence problem and interpreting the resulting ideal expressions as identity criteria of $\Im_T$ and $\Im_F$. Thus, the conception of New Semantics is only realized to its full extent if one only considers ab-expressions in terms of ab-symbols as identity criteria of $\Im_T$ and $\Im_F$ of *wff*s.

6.3.5 Construction of $\Im_T$ and $\Im_F$

This section describes a procedure to construct the $\Im_T$ and $\Im_F$ from a given ab-expression by iteration. This solves the problem of semantics (cf. p. 108). However, the mechanical construction of $\Im_T$ and $\Im_F$ starting from ab-expressions in general does not amount to a decision procedure. This is because of the following two reasons: (i) the process of generating models / counter-models does not terminate if I is infinite, and (ii) we start from ab-expressions (and not from ab-symbols). Thus, one cannot presume that, for example, tautologies, are identified by one common syntactical feature shared by all ab-expressions representing tautologies. In consequence, in the case of tautologies, the syntax of non-ideal ab-expressions generates inconsistent counter-models ad infinitum that are eliminated during the process of generation. In contrast, ab-symbols identify the empty class of counter-models by the empty class of b-pole-groups. Basing the process of generating counter-models upon ab-symbols would not even start to generate counter-models. Instead, it would generate the totality of models by systematically generating the totality of interpretations, $\Im_Q$. A decision procedure is only implied by the solution of the equivalence problem in terms of a procedure that generates *ab-symbols* from *wff*s by unambiguously minimizing ab-expressions. This procedure is independent of the procedure defined in this section. In the following, we describe how to construct the totality of models, $\Im_T$, and the totality of counter-models, $\Im_F$, of a *wff* A without deciding on single $\Im$ whether they are models or counter-models. Instead, those totalities are generated by *endless iteration*. We base this process on ab-expressions in general (and not only on ab-symbols) to verify through this procedure that $\Im_T$ and $\Im_F$ can be identified by the syntactical features of the ab-expressions.

The procedure of constructing $\Im_T$ and $\Im_F$ from an ab-expression consists of the following stages:

stage 1: construction of $\Im^*(\varphi)$, $\Im(t)$ and $\Im(\mathcal{J})$ from poles:

stage 1.1: construction of prefix-tables from predicate poles.

- **stage 1.2:** construction of $\Im^*(\varphi)$, $\Im(t)$, $\Im(\mathcal{J})$ with given I.

stage 2: construction of $\Im_T$ and $\Im_F$.

- **stage 2.1:** construction of the set union from $\Im^*(\varphi)$, $\Im(t)$, $\Im(\mathcal{J})$ of the single poles of a single pole-group.
- **stage 2.2:** construction of the $\Im_T$ and $\Im_F$ from the set unions.

These steps are defined and exemplified in the following.

Prefix-tables are tables that assign objects from I to the constituents of the pole's prefix, i.e. a name or a quantifier together with its forks. The head of a prefix-table contains the constituents of the prefix such that the n-th constituent (counted from left to right and top to bottom) is the head of column n of the prefix table.

Stage 1.1 is again decomposed in three stages.

stage 1.1: Prefix-tables must be generated for each complex pole.

1. Objects must be assigned to the universal quantifiers first.
   (a) Given that the number of universal quantifiers of a prefix-succession (cf. p. 263) is n, and that the number of objects from I is m, then the table must contain m^n combinations of the objects in the columns of the n universal quantifiers.
   (b) At first, the universal quantifiers of that prefix-succession that contain the most universal quantifiers are to be assigned with objects. The number of universal quantifiers of this prefix-succession defines the number of lines of the prefix-table. Thus, given that the number of those universal quantifiers is n, then the prefix-table has m^n lines.
   (c) The universal quantifiers of the remaining prefix-successions are subsequently assigned with objects from I. If the number of universal quantifiers of such a prefix-succession is k, one assigns the m^k combinations of objects to these universal quantifiers in the first m^k lines. Then these m^k possible combinations are repeated in the remaining lines until objects are assigned to these universal quantifiers in all m^n lines.
2. Next, objects are assigned to the existential quantifiers.

(a) If k universal quantifiers follow an existential quantifier in a prefix-succession, then $\exists\mu$ must be assigned with the same object in each of the m^k lines that contain the m^k combinations of those k universal quantifiers.

(b) The assignments of objects to some existential quantifier $\exists\mu$ is only restricted by the claim that, in certain lines, the same object is assigned to $\exists\mu$ according to 2(a). Besides this restriction, the assignments of objects to existential quantifiers may vary. *One* table merely contains *one* possibility for assigning objects to existential quantifiers. For one pole, all possible tables must be generated that contain all possible variations of assigning objects to existential quantifiers.

3. Finally, objects are to be assigned to names.

   (a) The same object must be assigned to every name in all lines of a table.

   (b) All tables are to be generated with all possible variations of assignments. If a table contains r names and I m objects, m^r possible variations of assignments are to be considered.

EXAMPLE 1: Given $I = \{c_1, c_2\}$ and the pole

$$\exists \left< \begin{array}{l} {}^{3}\ \forall_2\ \forall_1\ -a-F_{123} \\ {}_{3}\ \forall_2\ \exists_1\ -a-G_{123} \end{array} \right. .$$

Thus, a prefix table with five columns must be generated according to stage 1.1. The heads of the columns are the quantifiers $\exists < {}^{3}_{3}, \forall_2, \forall_2, \forall_1, \exists_1$. The pole contains two prefix-successions: $\exists < {}^{3}_{3}\forall_2\forall_1$ and $\exists < {}^{3}_{3}\forall_2\exists_1$. The first prefix-succession contains two universal quantifiers, while the second only has one universal quantifier. The number of lines is identical to the number of combinations of objects assigned to the universal quantifiers of the first prefix-succession. As this number is 2^2, the table has 2^2 lines. Stage 1.1.1(b) results in:

$\exists <{}^{3}_{3}$	$\forall_2$	$\forall_2$	$\forall_1$	$\exists_1$
	c_1		c_1	
	c_1		c_2	
	c_2		c_1	
	c_2		c_2	

To $\forall_2$ in the third column 2^1 objects are assigned, independently of the second and forth column. The 2^1 objects must be written down in the first 2^1 lines and then again in the next 2^1 lines. Thus, according to stage 1.1.1, the following result is obtained:

$\exists <{}^{3}_{3}$	$\forall_2$	$\forall_2$	$\forall_1$	$\exists_1$
	c_1	c_1	c_1	
	c_1	c_2	c_2	
	c_2	c_1	c_1	
	c_2	c_2	c_2	

According to stage 1.1.2, the same object is assigned to $\exists <{}^{3}_{3}$ in all lines of the table, whereas the objects assigned to $\exists_1$ may vary in each line. Let e_j be the number of lines with objects assigned to the j-th existential quantifier that may vary ($1 \leq e_j \leq m^n$)[14]. Then the number of tables with r existential quantifiers to be generated is $(m \cdot e_1) \cdot (m \cdot e_2) \cdot \ldots \cdot (m \cdot e_r)$. One of the possible $(2 \cdot 1) \cdot (2 \cdot 4) = 16$ tables for EXAMPLE 1 is the following:

---

[14] If the same object has to be assigned to an existential quantifier in all m^n lines, $e_j = 1$.

$\exists <{}^{3}_{3}$	$\forall_2$	$\forall_2$	$\forall_1$	$\exists_1$
c_1	c_1	c_1	c_1	c_1
c_1	c_1	c_2	c_2	c_2
c_1	c_2	c_1	c_1	c_1
c_1	c_2	c_2	c_2	c_2

EXAMPLE 2: Given $I = \{c_1, c_2\}$ and the pole (cf. EXAMPLE 2 on p. 268)

$$\exists \left< \begin{array}{l} ^{3}\ \exists_1\ \forall_2\ -a-F_{123} \\ _{3}\ \exists_1\ \forall_2\ -a-G_{123} \end{array} \right. .$$

According to stage 1.1, a table with five columns must be generated. The heads of the columns are $\exists <{}^{3}_{3}, \exists_1, \exists_1, \forall_2, \forall_2$. The pole contains two prefix-successions: $\exists <{}^{3}_{3}\exists_1\forall_2$ and $\exists <{}^{3}_{3}\exists_1\forall_2$. Each prefix-succession contains only one universal quantifier. Thus, only 2^1 combinations of objects must be assigned to each universal quantifier of each prefix-sequences. Thus, the table contains only 2^1 lines.

$\exists <{}^{3}_{3}$	$\exists_1$	$\exists_1$	$\forall_2$	$\forall_2$
			c_1	c_1
			c_2	c_2

According to stage 1.1.2, the same object must be assigned to each of the existential quantifiers $\exists <{}^{3}_{3}, \exists_1$ and $\exists_1$ in the 2^1 lines. Assigning c_1 to $\exists <{}^{3}_{3}$ in the first column, to $\exists_1$ in the second column, and c_2 to $\exists_1$ in the third column results in the following table:

$\exists <{}^{3}_{3}$	$\exists_1$	$\exists_1$	$\forall_2$	$\forall_2$
c_1	c_1	c_2	c_1	c_1
c_1	c_1	c_2	c_2	c_2

This is one of $(2\cdot 1)\cdot(2\cdot 1)\cdot(2\cdot 1) = 8$ tables differing by the objects assigned to the existential quantifiers.

EXAMPLE 3: Given $I = \{c_1, c_2\}$ and the pole

$$\begin{array}{l} \exists_1\,\forall_2 \\ \qquad\qquad \exists_1\,\forall_2\; \exists < \begin{array}{l} {}^{3}\; -a-F_{123} \\ {}_{3}\; -a-G_{123} \end{array}. \end{array}$$

This pole contains one prefix-succession with two universal quantifiers. According to stage 1.1.1, 2^2 combinations of objects are assigned to the universal quantifiers. Thus, the table has 2^2 lines.

$\exists_1$	$\forall_2$	$\exists_1$	$\forall_2$	$\exists < {}^{3}_{3}$
	c_1		c_1	
	c_1		c_2	
	c_2		c_1	
	c_2		c_2	

According to stage 1.1.2, the same object is assigned to $\exists_1$ of the first column in all four lines, whereas the same object is assigned to $\exists_1$ of the third column only in combination with 2^1 combinations of $\forall_2$ in the fourth column. That is, in lines 1-2 and in lines 3-4, the object assigned to $\exists_1$ in the third column may vary. Finally, the objects assigned to $\exists < {}^{3}_{3}$ may vary in each line. As I contains only two objects, it is not possible to select different objects in all four lines. Thus, one receives as one of $(2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 4) = 64$ possible tables:

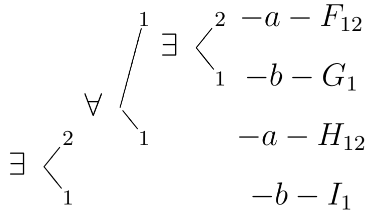
$\exists_1$	$\forall_2$	$\exists_1$	$\forall_2$	$\exists < {}^{3}_{3}$
c_1	c_1	c_1	c_1	c_2
c_1	c_1	c_1	c_2	c_1
c_1	c_2	c_2	c_1	c_2
c_1	c_2	c_2	c_2	c_1

To define stage 1.2, cf. p. 287, we must first specify classes of propositional functions not connected by open forks. We will denote such classes by "$\mathcal{C}(p)$".

$\mathcal{C}(p)$**-rule:** Generate the $\mathcal{C}(p)$ of a pole by applying the following rules moving from right to left:

1. If a pole contains only one propositional function, then this propositional function is the only element of a single $\mathcal{C}(p)$.
2. If propositional functions are connected by an open fork of the k-th column (cf. p. 239), then they are elements of different $\mathcal{C}(p)$.
3. If propositional function are connected by a closed fork of the k-th column, then they are elements of the same $\mathcal{C}(p)$.

Thus, the pole



contains two $\mathcal{C}(p)$: $\{a - F_{12}, b - G_1, b - I_1\}$ and $\{a - H_{12}, b - I_1\}$. These $\mathcal{C}(p)$ are generated as follows:

1. $a - F_{12}$ and $b - G_1$ are connected by a closed fork of the second column. Therefore, they are elements of the same $\mathcal{C}(p)$: $\{a - F_{12}, b - G_1\}$.

2. In the third column, $b - G_1$ and $a - H_{12}$, as well as $a - F_{12}$ and $a - H_{12}$, are connected by an open fork. Therefore, $a - H_{12}$ is an element of a further $\mathcal{C}(p)$: $\{a - F_{12}, b - G_1\}$, $\{a - H_{12}\}$.

3. The fourth column connects $b - I_1$ with $a - F_{12}$, with $b - G_1$, and with $a - H_{12}$ by a closed fork. Therefore, $b - I_1$ is an element of all those classes containing these elements: $\{a - F_{12}, b - G_1, b - I_1\}$, and $\{a - H_{12}, b - I_1\}$.

This procedure maps the application of DIS1. By translating a complex pole in a closed structure, eliminating all quantifiers, and converting the resulting expression in a disjunctive normal form by applying DIS1 from the inside to the outside, the translations of single disjuncts of the resulting disjunctive normal form are identical to the $\mathcal{C}(p)$.

In the following definition of stage 1.2, we refer to $\Im^*(\varphi)$. $\Im^*(\varphi)$ are interpretations of predicates containing tuples with, as well as without, superscribed bar, cf. p. 260. It is not presumed that *all* i^k-tuples must be contained in $\Im^*(\varphi)$. We use the term "redundant tuples" for all those tuples of the i^k tuples not contained in $\Im^*(\varphi)$ of a k-ary predicate. Whether or not they are contained in $\Im^*(\varphi)$ is not significant for determining whether $\Im$ is a model or counter-model.

Given

$$\begin{aligned} I = \quad & \{c_1,c_2\}, \\ \Im^*(F) = \quad & \{(c_1,c_2), \overline{(c_1,c_1)}, \overline{(c_2,c_2)}\} \end{aligned}$$

then

$\Im(F) = \{(c_1,c_2)\}, \overline{\Im}(F) = \{\overline{(c_1,c_1)}, \overline{(c_2,c_2)}\}$, while (c_2,c_1) is a redundant tuple.

By $\Im^*$, we mean interpretations with $\Im^*(\varphi)$ instead of $\Im(\varphi)$.

stage 1.2: $\Im^*$ of a pole must be generated by constructing $\Im^*(\varphi)$ and $\Im(t)$ from each single prefix table of a pole (1.-7.) in the case of predicate poles or by constructing $\Im(\mathcal{J})$ (8.) in the case of propositional poles.

1. All $\mathcal{C}(p)$ must be identified.

2. For every single line of a pole's prefix table, one $\mathcal{C}(p)$ must be selected. The corresponding objects of the respective line of the prefix table must be assigned to the argument positions of the propositional functions of the selected $\mathcal{C}(p)$. These assignments can be written down in a table with all propositional functions of the pole as heads of the columns.

3. Given several $\mathcal{C}(p)$, then the selection of objects for each line can be varied. All possible variations are to be generated in different tables. However, the following restriction is necessary to avoid constructing redundant tuples: if an assignment of objects can be selected that was already selected in a previous line, it must be selected again.

4. Each single resulting table determines a construction of $\Im^*(\varphi)$ and $\Im(t)$. A $\Im^*(\varphi)$ consists of all tuples assigned to those propositional functions containing φ. If the propositional function is a b-propositional function, the corresponding tuples must be written under a superscript bar.

5. If some $\Im^*(\varphi)$ contains contradictory tuples, that is, the same tuple, once with and once without a superscribed bar, the construction of the corresponding $\Im^*$ breaks down. In such a case, one has to go on with the construction of the next $\Im^*$. If a tuple is already contained in $\Im^*(\varphi)$, it must not be written down once more.

6. If c_i is assigned to a name t in a prefix table, and if this assignment is considered according to 1.2.2 in the table of propositional functions, then $\Im(t) = c_i$.

7. Each $\Im^*$ of a predicate pole consists of a presumed I together with the respective $\Im^*(\varphi)$ and $\Im(t)$, both generated according 1.-6.

8. If a pole is a propositional pole of the form $a - \mathcal{J}$, then $\Im^* = \Im(\mathcal{J}) = T$. If the propositional pole is of the form $b - \mathcal{J}$, then $\Im^* = \Im(\mathcal{J}) = F$.

EXAMPLE 1: The following pole of an a-pole-group

$$\exists \left\langle \begin{array}{l} {}^{3}\,\forall_2\,\forall_1 \; -a-F_{123} \\ {}_{3}\,\forall_2\,\exists_1 \; -a-G_{123} \end{array}\right.$$

contains only one $\mathcal{C}(p)$: $\{a - F_{123}, a - G_{123}\}$. One table according to stage 1.1 of this pole is the following, cf. EXAMPLE 1 on p. 282:

$\exists < \substack{3\\3}$	$\forall_2$	$\forall_2$	$\forall_1$	$\exists_1$
c_1	c_1	c_1	c_1	c_1
c_1	c_1	c_2	c_2	c_2
c_1	c_2	c_1	c_1	c_1
c_1	c_2	c_2	c_2	c_2

According to stage 1.2.2, the following tuples are assigned to the propositional functions in the respective lines:

$a - F_{123}$	$a - G_{123}$
$c_1c_1c_1$	$c_1c_1c_1$
$c_2c_1c_1$	$c_2c_2c_1$
$c_1c_2c_1$	$c_1c_1c_1$
$c_2c_2c_1$	$c_2c_2c_1$

As there is only one $\mathcal{C}(p)$, there is not need to consider variations of assignment to propositional functions. Thus, 16 different tables that assign tuples to propositional functions are received in accordance with the 16 possible tables from EXAMPLE 1, p. 282.

According to stage 1.2.4, the following $\Im^*(F)$ and $\Im^*(G)$ result from the aforementioned assignment of tuples to the propositional functions:

$\Im^*(F) = \{(c_1,c_1,c_1), (c_2,c_1,c_1), (c_1,c_2,c_1), (c_2,c_2,c_1)\}$,

$\Im^*(G) = \{(c_1,c_1,c_1), (c_2,c_2,c_1), (c_1,c_1,c_1), (c_2,c_2,c_1)\}$.

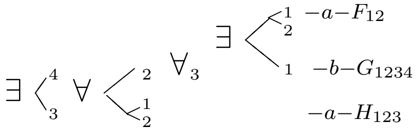
As the propositional functions are preceded by the a- and not the b-pole, $\Im^*(F)$ and $\Im^*(G)$ do not contain tuples with a superscribed bar. According to stage 1.2.5, repetitions of tuples have to be eliminated. According to 1.2.7, the domain I also has to be written down. Thus, according to stage 1.2, one obtains from the pole of EXAMPLE 1, p. 282 and its above mentioned prefix table, the following $\Im^*$ for a given domain $I = \{c_1, c_2\}$:

$I = \{c_1,c_2\}$,

$\Im^*(F) = \{(c_1,c_1,c_1), (c_2,c_1,c_1), (c_1,c_2,c_1), (c_2,c_2,c_1)\}$,

$\Im^*(G) = \{(c_1,c_1,c_1), (c_2,c_2,c_1)\}$.

EXAMPLE 2: Given the following pole



Assuming $I = \{c_1, c_2\}$, one prefix table of the pole is the following:

$\exists <^{4}_{3}$	$\forall <^{2}_{<^{1}_{2}}$	$\forall_3$	$\exists <^{<^{1}_{2}}_{1}$
c_1	c_1	c_1	c_1
c_1	c_1	c_2	c_2
c_1	c_2	c_1	c_1
c_1	c_2	c_2	c_2

As the pole contains two $\mathcal{C}(p)$, namely $\{a - F_{12}, b - G_{1234}\}$ and $\{a - H_{123}\}$, variations of assigning tuples to propositional functions must be considered according to stage 1.2.3. By considering the restriction mentioned in stage 1.2.3, these are the following four variations:

$a - F_{12}$	$b - G_{1234}$	$a - H_{123}$
c_1c_1	$c_1c_1c_1c_1$	
c_2c_2	$c_2c_1c_2c_1$	
c_1c_1	$c_1c_2c_1c_1$	
c_2c_2	$c_2c_2c_2c_1$	

$a - F_{12}$	$b - G_{1234}$	$a - H_{123}$
c_1c_1	$c_1c_1c_1c_1$	
c_2c_2	$c_2c_1c_2c_1$	
		$c_2c_2c_1$
		$c_2c_2c_1$

$a - F_{12}$	$b - G_{1234}$	$a - H_{123}$
		$c_1c_1c_1$
		$c_1c_1c_1$
c_1c_1	$c_1c_2c_1c_1$	
c_2c_2	$c_2c_2c_2c_1$	

$a - F_{12}$	$b - G_{1234}$	$a - H_{123}$
		$c_1c_1c_1$
		$c_1c_1c_1$
		$c_2c_2c_1$
		$c_2c_2c_1$

From these tables, the following $\Im^*$ are generated according to stages 1.2.4-1.2.7 for a given $I = \{c_1,c_2\}$:

1. I = $\{c_1,c_2\}$,
   $\Im^*(F)$ = $\{(c_1,c_1),(c_2,c_2)\}$,

$\Im^*(G) = \{\overline{(c_1,c_1,c_1,c_1)}, \overline{(c_2,c_1,c_2,c_1)}, \overline{(c_1,c_2,c_1,c_1)}, \overline{(c_2,c_2,c_2,c_1)}\}$.

2. $I = \{c_1,c_2\}$,
   $\Im^*(F) = \{(c_1,c_1), (c_2,c_2)\}$,
   $\Im^*(G) = \{\overline{(c_1,c_1,c_1,c_1)}, \overline{(c_2,c_1,c_2,c_1)}\}$,
   $\Im^*(H) = \{(c_2,c_2,c_1)\}$.
3. $I = \{c_1,c_2\}$,
   $\Im^*(F) = \{(c_1,c_1), (c_2,c_2)\}$,
   $\Im^*(G) = \{\overline{(c_1,c_2,c_1,c_1)}, \overline{(c_2,c_2,c_2,c_1)}\}$,
   $\Im^*(H) = \{(c_1,c_1,c_1)\}$.
4. $I = \{c_1,c_2\}$,
   $\Im^*(H) = \{(c_1,c_1,c_1), (c_2,c_2,c_1)\}$.

$\Im_T$ and $\Im_F$ are generated at stage 2, from the $\Im^*$ that are received at stage 1 from the poles.

stage 2: $\Im_T$ and $\Im_F$ are to be generated according to the following rules:

stage 2.1: All set unions from one $\Im^*$ of each pole of a pole-group with identical I must be generated. Thus, the following rules have to be considered:

1. If a $\Im^*(\varphi)$ of the set union under construction contains contradictory tuples, or if the set union under construction contains different interpretations of the same propositional variable or the same name, then one must break up the construction of the set union and go on generating the next set union.
2. $\Im(\mathcal{J})$, $\Im(t)$ and tuples of one $\Im^*(\varphi)$ must not be repeated.
3. If a resulting set union B of an a-pole-group / a b-pole-group is part of a resulting set union A of another a-pole-group / b-pole-group, then A must be eliminated.

stage 2.2: $\Im_T$ from the set unions assigned to a-pole-groups, and $\Im_F$ from the the set unions assigned to b-pole-groups are to be generated.

1. From each single set union assigned to a single pole-group, sets of $\Im^*$ must be generated. These are created by supplementing $\Im^*(\varphi)$ with all further, redundant possible tuples, with or without a superscribed bar, that can be generated from the objects of I. This must be done without generating contradictory tuples and without repeating tuples or $\Im^*$.
2. $\Im_T$ / $\Im_F$ result from these sets generated from the a-pole-groups / b-pole-groups if one eliminates tuples with superscribed bar and combines the resulting $\Im^*$ with all variations of all $\Im(\mathcal{J})$, $\Im(t)$, and $\Im(\varphi)$ that are not contained in the respective $\Im^*$.

EXAMPLE: Given the following a-pole-groups:

$$\begin{array}{l} a - \{\forall_1\ \ - a - F_1,\ \exists_1\ \ - a - G_1\}, \\ a - \{\exists_2\forall_1\ \ - a - H_{12}\}, \\ a - \left\{ \begin{array}{lll} & \forall <^{1}_{1} & \begin{array}{l} -a - F_1 \\ -a - H_{12} \end{array} \\ \exists <^{2}_{1} & & -a - G_1 \end{array} \right\}. \end{array}$$

For $I = \{c_1, c_2\}$ the following $\Im^*$ result from stage 1 for the single poles of the first a-pole-group:

For $\forall_1\ \ - a - F_1$:

1. $I = \{c_1, c_2\}$,
   $\Im^*(F) = \{c_1, c_2\}$.

For $\exists_1\ \ - a - G_1$:

1. $I = \{c_1, c_2\}$,
   $\Im^*(G) = \{c_1\}$.
2. $I = \{c_1, c_2\}$,
   $\Im^*(G) = \{c_2\}$.

In the first case, c_2, in the second case c_1 is a redundant 1-tuple of $\Im^*(G)$. According to stage 2.1, the $\Im^*$ that are assigned to the first a-pole-group result from the union of the $\Im^*$ of the two poles. Thus, the following $\Im^*$ of the first a-pole-group are obtained for $I = \{c_1, c_2\}$:

1. $I = \{c_1, c_2\}$,
   $\Im^*(F) = \{c_1, c_2\}$,
   $\Im^*(G) = \{c_1\}$.
2. $I = \{c_1, c_2\}$,
   $\Im^*(F) = \{c_1, c_2\}$,
   $\Im^*(G) = \{c_2\}$.

According to stage 1, the following $\Im^*$ are received for the pole $\exists_2\forall_1\ -a-H_{12}$ of the second a-pole-group:

1. $I = \{c_1, c_2\}$,
   $\Im^*(H) = \{(c_1,c_1), (c_2,c_1)\}$.
2. $I = \{c_1, c_2\}$,
   $\Im^*(H) = \{(c_1,c_2), (c_2,c_2)\}$.

As the second a-pole-group contains only one pole, these $\Im^*$ are identical to the $\Im^*$ of the second pole-group.

According to stage 1, the following $\Im^*$ are received for the pole

$$\exists \begin{array}{l} {}^{2}\ \forall \begin{array}{l} {}^{1}\ -a-F_1 \\ {}_{1}\ -a-H_{12} \end{array} \\ {}_{1}\qquad\quad -a-G_1 \end{array}$$

of the third a-pole-group:

1. $I = \{c_1, c_2\}$,
   $\Im^*(F) = \{c_1, c_2\}$.
2. $I = \{c_1, c_2\}$,
   $\Im^*(F) = \{c_1\}$,
   $\Im^*(G) = \{c_1\}$,

$\Im^*(H) = \{(c_2,c_1)\}$.

3. $I = \{c_1, c_2\}$,
   $\Im^*(F) = \{c_1\}$,
   $\Im^*(G) = \{c_2\}$,
   $\Im^*(H) = \{(c_2,c_2)\}$.

4. $I = \{c_1, c_2\}$,
   $\Im^*(F) = \{c_2\}$,
   $\Im^*(G) = \{c_1\}$,
   $\Im^*(H) = \{(c_1,c_1)\}$.

5. $I = \{c_1, c_2\}$,
   $\Im^*(F) = \{c_2\}$,
   $\Im^*(G) = \{c_2\}$,
   $\Im^*(H) = \{(c_1,c_2)\}$.

6. $I = \{c_1, c_2\}$,
   $\Im^*(G) = \{c_1\}$,
   $\Im^*(H) = \{(c_1,c_1),(c_2,c_1)\}$.

7. $I = \{c_1, c_2\}$,
   $\Im^*(G) = \{c_2\}$,
   $\Im^*(H) = \{(c_1,c_2),(c_2,c_2)\}$.

As the third a-pole-group does not contain only one pole, these $\Im^*$ are identical with the $\Im^*$ of the third pole-group.

The first $\Im^*$ of the third a-pole-group is a subset of the two $\Im^*$ of the first a-pole-group. Therefore, these two $\Im^*$ are eliminated according to stage 2.1.3. The two $\Im^*$ of the second a-pole-group are subsets of the two last $\Im^*$ of the third a-pole-group. Therefore, these two $\Im^*$ of the last a-pole-group are eliminated as well. Consequently, the following seven $\Im^*$ with domain $I = \{c_1, c_2\}$ result, according to stage 2.1: the two $\Im^*$ of the second a-pole-group and the first five $\Im^*$ of the third a-pole-group. According to stage 2.2, $\Im_T$ is generated from these seven $\Im^*$. According to stage 2.2.1, tuples with and without a superscribed bar must be added to the $\Im^*$. These tuples are generated from c_1 and c_2 . However, contradictory tuples, repeated tuples, or repeated $\Im^*$ must not occur. Thus, for example, one receives for the first $\Im^*$, $I = \{c_1, c_2\}$; $\Im^*(H) = \{(c_1{,}c_1), (c_2{,}c_1)\}$, of the second a-pole-group the following variations:

1. $I = \{c_1{,}c_2\}$,
   $\Im^*(H) = \{(c_1{,}c_1), (c_2{,}c_1), (c_1{,}c_2), (c_2{,}c_2)\}$.
2. $I = \{c_1{,}c_2\}$,
   $\Im^*(H) = \{(c_1{,}c_1), (c_2{,}c_1), (c_1{,}c_2), \overline{(c_2{,}c_2)}\}$.
3. $I = \{c_1{,}c_2\}$,
   $\Im^*(H) = \{(c_1{,}c_1), (c_2{,}c_1), \overline{(c_1{,}c_2)}, (c_2{,}c_2)\}$.
4. $I = \{c_1{,}c_2\}$,
   $\Im^*(H) = \{(c_1{,}c_1), (c_2{,}c_1), \overline{(c_1{,}c_2)}, \overline{(c_2{,}c_2)}\}$.

According to stage 2.2.2, $\Im_T$ results from those $\Im^*$ that result from stage 2.2.1. They result from those $\Im^*$ by eliminating the tuples with a superscribed bar and combining the respective results with all interpretations of propositional variables, names and predicates that are not contained in the respective $\Im^*$.

The first $\Im^*$ resulting from stage 1.2, namely $\Im^*(H) = \{(c_1{,}c_1), (c_2{,}c_1)\}$, is the constant feature of the four $\Im^*$ of the second a-pole-group, which result from stage 2.2.1. The variations of the supplemented tuples are insignificant features of this class of models. The first $\Im^*$ resulting from stage 1.2 is itself an element of a class of $\Im^*$ that varies with respect to the object at the second position of the 2-tuples of $\Im^*(H)$, which is combined with all objects at the 1. position of the 2-tuples of $\Im^*(H)$. These variations, again, are part of different $\Im^*$ with varying I, and they are part of the variations stemming from varying all other interpretations of propositional variables, names, and predicates. The common feature of all these variations is the fact that some object at the 2. position of the 2-tuples of $\Im^*(H)$ is combined with all objects at the 1. position of the 2-tuples of $\Im^*(H)$. This feature is identified by the pole of the second a-pole-group. It is the structural feature of one of three subclasses of $\Im_T$.

ab-expressions identify constant, structural features of classes of models and counter-models. The totality of $\Im_T$ and $\Im_F$ result from systematically varying insignificant features while the structural features remain constant. These structural features are represented by the finite ab-expressions. The potentially infinite elements of $\Im_T$ and $\Im_F$ can be constructed by systematically varying insignificant parts. By identifying structural features of classes of models and counter-models by ab-expressions, New Semantics is superior to classical semantics because it makes possible a systematic construction of $\Im_T$ and $\Im_F$.

6.3.5.1 Searching for Models

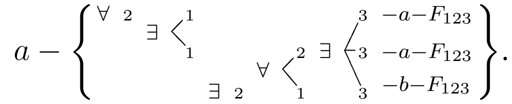
ab-expression have a practical significance with respect to searching for models of a *wff*. This will be demonstrated by the two examples that follow.

EXAMPLE 1: Given the *wff*

$$\forall x_1 \exists y_1 \exists y_2 \forall x_2 \exists y_3 (F y_1 x_1 y_3 \wedge F y_1 x_2 y_3 \wedge \neg F x_2 y_2 y_3). \tag{6.28}$$

The satisfiability of (6.28) is in question. *Prima facie* this question cannot be answered. There is no model with $I = \{c_1\}$ or $I = \{c_1, c_2\}$. Examining the $\Im$ with $I = \{c_1, c_2, c_3\}$ is not practical nor manageable, as there are $2^{3^3} = 2^{27} = 134\,217\,728$ $\Im$ of (6.28).[15] Furthermore, the negation of (6.28) is not derivable as a theorem because (6.28) has a model. Thus, one must construct a model with a domain containing at least three objects to prove the satisfiability of (6.28).[16]

(6.28) is a closed structure. Its a-pole-group is the following:



From stage 1.1, the following table results with $I = \{c_1, c_2, c_3\}$:

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[15] A common programme for examining interpretations evaluates about 200,000 interpretations of (6.28) within 10 hours, cf. the programme mentioned on p. 259. Within a realistic amount of time, such a programme is not able to prove the satisfiability of (6.28).

[16] The theorem prover of Gottschall does prove the negation of the formula, cf. http://logik.phl.univie.ac.at/ chris/formular-ableitung.html. However, it contains a mistake in line 5, cf. http://logik.phl.univie.ac.at/ chris/gateway/pl-beispiele.html. Only by successfully searching a model of (6.28) was it possible to demonstrate with certainty that any proof of ¬(6.28) as a theorem must be mistaken.

$\forall_2$	$\exists < \substack{1\\1}$	$\exists_2$	$\forall < \substack{2\\1}$	$\exists - \substack{3\\3\\3}$
c_1	c_1	c_3	c_1	c_2
c_1	c_1	c_3	c_2	c_2
c_1	c_1	c_3	c_3	c_1
c_2	c_2	c_1	c_1	c_3
c_2	c_2	c_1	c_2	c_1
c_2	c_2	c_1	c_3	c_1
c_3	c_3	c_2	c_1	c_3
c_3	c_3	c_2	c_2	c_2
c_3	c_3	c_2	c_3	c_3

Different objects are assigned to existential quantifiers in this table if possible. From stage 1.2.3, the following table results:

$a - F_{123}$	$a - F_{123}$	$b - F_{123}$
$c_1c_1c_2$	$c_1c_1c_2$	$c_1c_3c_2$
$c_1c_1c_2$	$c_1c_2c_2$	$c_2c_3c_2$
$c_1c_1c_1$	$c_1c_3c_1$	$c_3c_3c_1$
$c_2c_2c_3$	$c_2c_1c_3$	$c_1c_1c_3$
$c_2c_2c_1$	$c_2c_2c_1$	$c_2c_1c_1$
$c_2c_2c_1$	$c_2c_3c_1$	$c_3c_1c_1$
$c_3c_3c_3$	$c_3c_1c_3$	$c_1c_2c_3$
$c_3c_3c_2$	$c_3c_2c_2$	$c_2c_2c_2$
$c_3c_3c_3$	$c_3c_3c_3$	$c_3c_2c_3$

From stage 1.2, the following $\Im^*$ results:

$$\begin{aligned} I = \quad & \{c_1, c_2, c_3\}, \\ \Im^*(F) = \quad & \{(c_1,c_1,c_2), \overline{(c_1,c_3,c_2)}, (c_1,c_2,c_2), \overline{(c_2,c_3,c_2)}, (c_1,c_1,c_1), (c_1,c_3,c_1), \overline{(c_3,c_3,c_1)}, \\ & (c_2,c_2,c_3), (c_2,c_1,c_3), \overline{(c_1,c_1,c_3)}, (c_2,c_2,c_1), \overline{(c_2,c_1,c_1)}, (c_2,c_3,c_1), \overline{(c_3,c_1,c_1)}, \\ & (c_3,c_3,c_3), (c_3,c_1,c_3), \overline{(c_1,c_2,c_3)}, (c_3,c_3,c_2), (c_3,c_2,c_2), \overline{(c_2,c_2,c_2)}, \overline{(c_3,c_2,c_3)}\}. \end{aligned}$$

This $\Im^*$ does not contain any contradictory tuples. It specifies a class of models, namely the class of models varying in respect to the six redundant tuples[17] that are not contained in this $\Im^*$ with or without a superscript bar. Thus, a model of (6.28) results from eliminating all tuples with a superscript bar in the aforementioned $\Im^*$. Instead of examining $134\,217\,728$ $\Im$ one by one, a model can be generated directly, and with immeasurably less effort for (6.28), by the defined procedure.

EXAMPLE 2: Bernays and Hilbert mention the following system of axioms in their *Foundations of Mathematics* (Bernays (1968), p. 14.):

$$\begin{gathered} \forall x \exists y Fxy, \\ \forall x \forall y \forall z (Fxy \land Fyz \rightarrow Fxz), \\ \forall x \neg Fxx. \end{gathered}$$

This axiomatic system has no model with finite I (cf. Bernays (1968), p. 14 and, in detail, Lampert (2005a), p. 233-236). Nevertheless it is satisfiable. This, Bernays and Hilbert make evident by the following standard interpretation (Bernays (1968), p. 14 (author's translation)):

> [...] we see that the axioms are not satisfiable within a finite domain. However, we can easily provide a model with an infinite domain: We take as individuals the integers, and interpret $F(x,y)$ as "x is smaller than y". Then, it arises immediately that all three axioms are satisfied.

This interpretation is not the result of a systematic search for a model, but is based on "mathematical imagination" (cf. Kreisel (1987), p. 509). In the following, we show how to apply the described method of generating models from ab-expression to systematically generate models with an infinite domain. However, it is not maintained that the procedure is a proof-procedure that proves the consistency of the axiomatic systems in a finite number of steps. This can only be done by generating its ab-symbol comprising a-pole-groups. Instead, the intention is to show that ab-expressions allow one to systematically construct $\Im_T$ and $\Im_F$ such that determining a model is not a fortunate coincidence of deciding

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[17] These are the tuples $(c_1,c_2,c_1), (c_1,c_3,c_3), (c_2,c_1,c_2), (c_2,c_3,c_3), (c_3,c_1,c_2), (c_3,c_2,c_1)$. The aforementioned $\Im^*$ contains 21 triples. With the three objects c_1, c_2, c_3 $3^3 = 27$, triples can be generated. Thus, $27 - 21 = 6$ tuples may vary.

upon some arbitrary given $\Im$ but the result of a methodical construction. "Searching" for a model within an infinite domain is only possible if this search is based on symbolizing properties of ab-expressions paving the way for specifying the structure of models.

A_T of (6.29) is:

$$\forall x_1 \exists y_1 F x_1 y_1 \wedge \forall x_2 \forall x_3 (\neg F x_2 x_3 \vee \forall x_4 (\neg F x_3 x_4 \ \vee \ F x_2 x_4)) \wedge \forall x_5 \neg F x_5 x_5. \quad (6.29)$$

The translation into an a-pole-group is:[18]

$$a - \left\{ \forall \ {}_1 \ \exists \ {}_2 \ -a-F_{12}, \quad \forall <^1_1 \ -b-F_{12}, \ \forall \begin{array}{llllll} {}^{1}\diagup & \forall & <^{2} & & & -b-F_{12} \\ & & \quad {}_{1} & & {}^{2} & -b-F_{12} \\ {}_{1}\diagdown & & & \forall & <_{2} & -a-F_{12} \end{array} \right\}.$$

According to stage 1, the following $\Im^*$ results for the first pole, $\forall_1 \exists_2 \ - a - Fxy$, if one assigns different objects to the existential quantifier where possible:

$$\begin{array}{ll} I = & \{c_1, c_2, \ldots\}, \\ \Im^*(F) = & \{(c_1,c_2), (c_2,c_3), (c_3,c_4), \ldots\}. \end{array}$$

The indices of the first and the second elements increase by one in every subsequent pair.

According to stage 1, the following $\Im^*$ results for the second pole $\forall <^{1}_{2} - b - F_{12}$:

$$\begin{array}{ll} I = & \{c_1, c_2, \ldots\}, \\ \Im^*(F) = & \{\overline{(c_1,c_1)}, \overline{(c_2,c_2)}, \overline{(c_3,c_3)}, \ldots\}. \end{array}$$

The indices of the first and the second element increase by one in every following pair.

In constructing the following tables generated from the third pole, we use the following conventions to symbolize combinations of infinite series of elements. $c_i \ldots c_i$ stands for an infinite repetition of the same element. $c_1 \ldots c_\infty$ stands for an infinite series of elements, starting with c_1 and then applying the operation of adding 1 to generate the next element. ∞ is not meant to be some number or extension. Instead, it indicates iterative application of the operation of adding 1. Furthermore, we allow for combining and iterating these series. The criterion for

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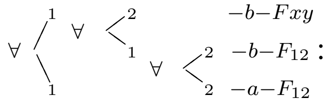
[18] The order of the poles is arbitrary.

well-formed series and their composition is that the single series, as well as their composition, are definable by operations; one must be able to "recognize a law" in the construction of the tables.

According to stage 1.1, the following prefix table results for the third pole



$\forall <^{1}_{1}$	$\forall <^{2}_{1}$	$\forall <^{2}_{2}$
c_1	c_1	c_1
	⋮	⋮
	c_1	c_∞
⋮	⋮	⋮
	c_∞	c_1
	⋮	⋮
c_1	c_∞	c_∞
⋮	⋮	⋮
c_∞	c_1	c_1
	⋮	⋮
	c_1	c_∞
		c_1
⋮	⋮	⋮
		c_∞
	c_∞	c_1
	⋮	⋮
c_∞	c_∞	c_∞

The third pole contains three propositional functions that are connected by an open fork. According to stage 1.2, a tuple from every line must be assigned to only one of the three propositional functions. Without selecting only one of the three propositional functions, the following tuples are assigned to the propositional functions:

$b - F_{12}$	$b - F_{12}$	$a - F_{12}$
c_1,c_1	c_1,c_1	c_1,c_1
$\vdots$	$\vdots$	$\vdots$
c_1,c_1	c_1,c_∞	c_1,c_∞
$\vdots$	$\vdots$	$\vdots$
c_1, c_∞	c_∞, c_1	c_1, c_1
$\vdots$	$\vdots$	$\vdots$
c_1,c_∞	c_∞,c_∞	c_1,c_∞
$\vdots$	$\vdots$	$\vdots$
c_∞,c_1	c_1,c_1	c_∞,c_1
$\vdots$	$\vdots$	$\vdots$
c_∞,c_1	c_1,c_∞	c_∞,c_∞
$\vdots$	$\vdots$	$\vdots$
c_∞,c_∞	c_∞,c_1	c_∞,c_1
$\vdots$	$\vdots$	$\vdots$
c_∞,c_∞	c_∞,c_∞	c_∞,c_∞

According to stage 1.2.2, only one tuple must be selected in each line. This selection can be confined by considering the $\Im^*$ of the first two poles. According to stage 2, the set union of each of the $\Im^*$ of the three poles must be generated. To avoid tuples that are not essential for a model, tuples should be selected that are elements of the $\Im^*$ of the first or second pole, if possible. That is to say, we select tuples of the type (c_i,c_{i+1}) for the third a-propositional function, and we select tuples of type (c_i,c_i) for the first two b-propositional functions. Furthermore, contradictory tuples must not be generated. This leads to a further confinement of the selection of tuples. If one of the two b-propositional functions is assigned a tuple of type (c_i,c_{i+1}), then this must not be selected. Likewise, if the a-propositional function is assigned a tuple of type (c_i,c_i), this must not be selected, either. Tuples of type (c_j,c_i) $(j > i)$ must be assigned to b-propositional functions, whereas tuples of type (c_i,c_j) $(i < j)$ must be assigned to the a-propositional function. Thus, according to stage 1, the following $\Im^*$, which obviously satisfies a certain structure definable by operations, results for the third pole:

$$\begin{aligned} I = &\quad \{c_1, c_2, c_3, \ldots\},\\ \Im^*(F) = &\quad \{\overline{(c_1,c_1)}, (c_1,c_2), (c_1,c_3),(c_1,c_4), \ldots\\ &\quad (\overline{(c_2,c_1)}, \overline{(c_2,c_2)}, (c_2,c_3), (c_2,c_4), \ldots\\ &\quad (\overline{(c_3,c_1)}, \overline{(c_3,c_2)}, \overline{(c_3,c_3)}, (c_3,c_4), \ldots\\ &\qquad\qquad\qquad \ddots \qquad\qquad\qquad \}. \end{aligned}$$

As this $\Im^*$ contains the $\Im^*$ of the two first poles, it is identical to a set union of the $\Im^*$ specifying a model of (6.29). No further tuples can be supplemented to this $\Im^*$ without generating contradictory tuples. This is because if one eliminates the subscribed bar, then all possible tuples are mentioned. Thus, no $\Im(F)$ can be generated by systematically varying inessential tuples from this $\Im^*$. The constructed $\Im^*(F)$ denotes a $\Im(F)$, which contains only those tuples without a superscribed bar. This specifies a model of (6.29), and thus shows that (6.29) is satisfiable.

Varying the objects assigned to the existential quantifier in the first pole would result in variations of the $\Im^*$ of the a-pole-group. From this, $\Im^*$ result for which single tuples are redundant. However, the *constant* features of all models of (6.29) are articulated by the paraphrase of the a-pole-group:

- An $\Im$ of the axiom system

$$\begin{gathered} \forall x \exists y Fxy, \\ \forall x \forall y \forall z (Fxy \wedge Fyz \;\rightarrow\; Fxz), \\ \forall x \neg Fxx \end{gathered}$$

  is a model iff

- all objects at the 1. position of the 2-tuples of $\Im(F)$ combined with some object at the 2. position of the 2-tuple of $\Im(F)$ satisfy $\Im^*(F)$;
- all objects, the same at the 1. position of the 2-tuples of $\overline{\Im}(F)$ and at the 2. position of the the 2-tuples of $\overline{\Im}(F)$, satisfy $\Im^*(F)$; and
- all objects, distributed among the 1. position of the 2-tuples of $\overline{\Im}(F)$ and the 1. position of the 2-tuples of $\overline{\Im}(F)$, combined with all objects, distributed among the 2. position of the 2-tuples of $\overline{\Im}(F)$ and the 1. position of the 2-tuples of $\overline{\Im}(F)$, combined with all objects, distributed among the 2. position of the 2-tuples of $\overline{\Im}(F)$ and the 2. position of the 2-tuples of $\Im(F)$, satisfy $\Im^*(F)$.

6.4 System of implications between $\bigvee\bigwedge cs$

In this section, we define a calculus to generate the system of implications between $\bigvee\bigwedge cs$. Any logical derivation within the system of $\bigvee\bigwedge cs$ must be performed by *operations* in Wittgenstein's sense. All implications between $\bigvee\bigwedge cs$ must be reducible to systematically varying syntactic properties of $\bigvee\bigwedge cs$. We show that such a reduction is possible by a calculus consisting of 13 rules of implication which all specify some syntactic variation that constitutes a relation of implication without any further constraints. These rules generalize the 13 rules of implication of elementary predicate logic to apply them to molecular logic. In section 6.4.1 we define the calculus, in section 6.4.2 we prove its correctness and completeness.

We stick to $\bigvee\bigwedge cs$ instead of pole-groups to make use of familiar derivation rules and equivalent transformations. However, as $\bigvee\bigwedge cs$ and pole-groups are interconvertible, it is likewise possible to refer to pole-groups. Thus, we also refer to the pole-group notation when this is convenient.

We call two $\bigvee\bigwedge cs$ identical iff their translation in the pole-group notation is identical. Thus, differences of equivalent $\bigvee\bigwedge cs$ concerning the type of variables, the order of disjuncts and conjuncts or the order of existential quantifiers in a sequence of existential quantifiers or the order of universal quantifiers in a sequence of universal quantifiers are to be ignored.

6.4.1 Rules of implication

In analogy to the calculus for elementary poles, in this section, we define rules of implications in terms of operations identifying implications between $\bigvee\bigwedge cs$.

These rules are generalizations of the 13 rules in elementary logic: one rule concerning the ordering of quantifiers; three pairs of rules concerning the elimination of universal quantifiers and the introduction of existential quantifiers; and three pairs of rules referring to the introduction and elimination of conjunction and disjunction, including the introduction of tautologies or elimination of contradictions, cf. table 6.9. All these rules define a certain syntactic variation that is sufficient to justify that a certain $\bigvee\bigwedge cs\ A$ implies another $\bigvee\bigwedge cs\ B$.

$\exists\forall Ex$: $\exists\mu\ldots\forall\nu \vdash \forall\nu\ldots\exists\mu$	
$\forall E1$: $\forall\mu A(\mu) \vdash A(\mu/t)$	$\exists I1$: $A(t) \vdash \exists\mu A(t,t/\mu)$
$\forall E2$: $\forall\mu\ldots\forall\nu A(\mu,\nu) \vdash \forall\mu A(\mu,\nu/\mu)$	$\exists I2$: $\exists\mu A(\mu) \vdash \exists\mu\ldots\exists\nu A(\mu,\mu/\nu)$
$\forall E3$: $\exists\mu\ldots\forall\nu A(\mu,\nu) \vdash \exists\mu A(\nu/\mu)$	$\exists I3$: $\forall\mu \vdash \forall\mu\ldots\exists\nu A(\mu,\mu/\nu)$
$\wedge E$: $\bigwedge A, B \vdash A$	$\vee I$: $A \vdash \bigvee A, B$
$\wedge I$: $A \vdash \bigwedge A, A$	$\vee E$: $\bigvee A, A \vdash A$
$\top I$: $A \vdash \bigwedge A, \top$	$\bot E$: $\bigvee A, \bot \vdash A$

Table 6.9: Rules of implication

As in the case of elementary logic, the rules specify only those symbolizing properties that vary left and right of $\vdash$. They are valid wherever the respective properties occur in the $\bigvee\bigwedge cs$.[19]

We write $\bigwedge A, B$ and $\bigvee A, B$ instead of $A \wedge B$ and $A \vee B$ to stress that the ordering of conjuncts and disjuncts in conjunctions or disjunctions with n elements is insignificant. We merely presume that A and B in $\bigwedge A, B$ are conjuncts in a conjunction of n conjuncts, and $\bigvee A, B$ are disjuncts in a disjunction of n disjuncts.

$\wedge I$, $\top$ I, $\vee E$, $\bot$ E are, in fact, rules of equivalence. Due to $\wedge E$ and $\vee I$, they also hold for the direction from right to left. We call the defined direction from left to right the "critical direction" and call the mentioned four rules "the four rules of equivalence".

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[19] One has to keep in mind that $\bigvee\bigwedge cs$ are formulae in negative normal form (NNF) that merely contain $\wedge$ and $\vee$ as dyadic sentential connectives. Furthermore, we presume that each variable occurring in a $\bigvee\bigwedge cs$ is bound by only one quantifier (cf. p. 247).

Before paraphrasing the rules, we establish some definitions used in the paraphrasing of the rules.

DEFINITIONS:

∃-formula: ∃-formulae are formulae of elementary predicate logic consisting of a sequence of existential quantifiers preceding an atomic expression. This atomic expression only contains existential variables, each of which only occur once. We abbreviate ∃-formulae as follows:

$$\exists v_1 \ldots \exists v_n [\neg] \varphi(v_1 \ldots v_n)$$

n might be 0. In this case, the ∃-formula is a propositional literal.

∀-formula: ∀-formulae are formulae of elementary predicate logic that are conjunctions of formulae consisting of a sequence of universal quantifiers preceding an atomic expression. This atomic expression only contains universal variables, each of which occur only once. We abbreviate ∀-formulae as follows:

$$\bigwedge_{1}^{k} \forall v_{1_1} \ldots \forall v_{1_m} [\neg] \psi_1, \ldots, \forall v_{k_1} \ldots \forall_{k_o} [\neg] \psi_k$$

m and o might be 0. In this case, the conjuncts of the ∀-formula are in the form of propositional literals.

∃-formulae correspond to ∃-poles, ∀-formulae correspond to a pole-group of ∀-poles (cf. p. 174).

∃-contradiction: ∃-contradictions are formulae consisting of a sequence of existential quantifiers preceding a conjunction of two contradictory atomic expressions. These atomic expressions only contain existential variables, each of which not occurring twice in one atomic expression. We abbreviate ∃-contradictions as follows:

$$\exists v_1 \ldots \exists v_n \bigwedge \varphi(v_1 \ldots v_n), \neg\varphi(v_1 \ldots v_n)$$

n might be 0. In this case, the ∃-contradiction is a propositional contradiction of the form $\bigwedge A, \neg A$.

$\forall$-tautology: $\forall$-tautologies are formulae consisting of a sequence of universal quantifiers preceding a disjunction of two contradictory atomic expressions. These atomic expressions only contain universal variables, each of which not occurring twice in one atomic expression. We abbreviate $\forall$-tautologies as follows:

$$\forall v_1 \ldots \forall v_n \bigvee \varphi(v_1 \ldots v_n), \neg\varphi(v_1 \ldots v_n)$$

n might be 0. In this case, the $\forall$-tautology is of the form $\bigvee A, \neg A$.

In the paraphrases of the rules, we distinguish between *general* and *limited applications* of the rules. General applications of the rules comprise their limited application. The results of a general application of the rules are also achievable by the limited applications of a rule, plus, if necessary, further applications of the rules of implication despite from the rules $\wedge I$ and $\top I$. We abstain from these two rules of equivalence because by these two rules implied $\bigvee \bigwedge cs$ can be added endlessly. For example, it is possible to generate the following path of derivation by applying our rules of implication: $\forall xFx \vdash \forall xFx \wedge \forall xFx \vdash Fa \wedge \forall xFx \vdash Fa \wedge Fb \vdash Fa$. Thus, if we would not abstain from $\wedge I$, we could never identify any application of $\forall E1$ that results in a $\bigvee \bigwedge cs$ that could not be achieved by further applications of the rules of implication involving $\wedge I$ and $\wedge E$. Similar consideration hold for $\top I$. It might be convenient to apply the rules in a general way to identify implications between two $\bigvee \bigwedge cs$. However, it suffices to apply the rules in a limited way to identify all relations of implication because applying the rules in a general way can always be replaced by applying them in a limited way to achieve the same results.

PARAPHRASE:

$\exists\forall$-exchange rule ($\exists\forall Ex$): A $\bigvee \bigwedge cs$ $\mathcal{A}$ implies a $\bigvee \bigwedge cs$ $\mathcal{B}$, if $\mathcal{B}$ is generated from $\mathcal{A}$ by exchanging the order of quantifiers in a cs. In the antecedent, a universal quantifier $\forall\nu$ is in the scope of an existential quantifier $\exists\mu$. In the consequent, the existential quantifier $\exists\mu$ is in the scope of the universal quantifier $\forall\nu$:

$$\exists\mu \ldots \forall\nu \vdash \forall\mu \ldots \exists\nu$$

The resulting expression must be converted to a $\bigvee \bigwedge cs$ to achieve $\mathcal{B}$.
It suffices to limit the application of $\exists\forall Ex$ to $\forall\nu$ directly succeeding $\exists\mu$ in the antecedent and directly preceding $\exists\mu$ in the consequent:

$$\exists\mu\forall\nu \vdash \forall\mu\exists\nu$$

If necessary, this limited application of $\exists\forall Ex$ must be prepared by $\forall Ex$, $\exists Ex$ and PN1-4[20] such that this limited condition of the application of $\exists\forall Ex$ is satisfied.

universal quantifier elimination 1 ($\forall E1$)**:** A $\bigvee\bigwedge cs\ \mathcal{A}$ implies a $\bigvee\bigwedge cs\ \mathcal{B}$, if $\mathcal{B}$ is generated from $\mathcal{A}$ by eliminating a universal quantifier $\forall\mu$ and replacing the variable μ in all its occurrences by a name t:

$$\forall\mu A(\mu) \vdash A(\mu/t)$$

The resulting expression must be converted to a $\bigvee\bigwedge cs$ to achieve $\mathcal{B}$.
It suffices to limit the application of $\forall E1$ to universal quantifiers preceding a cs in $\mathcal{A}$.

universal quantifier elimination 2 ($\forall E2$)**:** A $\bigvee\bigwedge cs\ \mathcal{A}$ implies a $\bigvee\bigwedge cs\ \mathcal{B}$, if $\mathcal{B}$ is generated from $\mathcal{A}$, by eliminating a universal quantifier $\forall\nu$ that is in the scope of a universal quantifier $\forall\mu$, and replacing ν in all its occurrences by μ:

$$\forall\mu\ldots\forall\nu A(\mu,\nu) \vdash \forall\mu A(\mu,\nu/\mu)$$

The resulting expression must be converted to a $\bigvee\bigwedge cs$ to achieve $\mathcal{B}$. If necessary, the application of $\forall E2$ has to be prepared by PN1-8 and $\forall Ex$, to ensure that $\forall\nu$ is in the scope of $\forall\mu$.
It suffices to limit the application of $\forall E2$ to $\forall\nu$ directly following $\forall\mu$:

$$\forall\mu\forall\nu A(\mu,\nu) \vdash \forall\mu A(\mu,\nu/\mu)$$

The preparation of this limited application can be limited to the following two cases:

- $\forall\mu$ and $\forall\nu$ both occur in a universal quantifier sequence in the corresponding pole. In this case it suffices to apply PN1-4 and $\forall Ex$ to ensure that $\forall\nu$ directly follows $\forall\mu$ (cf. footnote 20).

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[20] $\exists Ex$ is needed to change the order of existential quantifiers in a sequence of existential quantifiers. $\forall Ex$ and PN1-4 are needed to change the order of universal quantifiers, $\wedge$ and $\vee$, in the logical hierarchy of the formula. Unlike the general application, the limited application does not allow for existential quantifiers occurring between these universal quantifiers, $\wedge$ and $\vee$. In the corresponding pole of the ab-notation $\forall\mu$ is part of a sequence of universal quantifiers directly following a sequence of existential quantifiers containing $\exists\mu$. In the following we refer to “universal quantifier sequences” if the universal quantifiers correspond to a sequence of universal quantifiers in the corresponding pole. Likewise, we refer to “existential quantifier sequences”. Thus, we abstain from $\wedge$ or $\vee$ occurring between the quantifiers in the hierarchy of the formulae in these cases.

- $\forall\nu$ does not occur in the scope of $\forall\mu$ and v.v. In this case, it suffices to apply the PN-rule to prepare the application of $\forall E2$ (cf. below p. 313).

However, if $\forall\mu$ and $\forall\nu$ occur in different conjuncts, applying $\forall E2$ does not result in different $\bigvee\bigwedge cs$. The reason for this is that converting the expression back to a $\bigvee\bigwedge cs$ countermands the elimination of $\forall\nu$ due to the application of PN9 in $\bigvee\bigwedge cs$-R. Thus, the application of $\forall E2$ is useless in this case. We label the application of $\forall E2$ without converting the resulting expression back to a $\bigvee\bigwedge cs$ in case $\forall\mu$ and $\forall\nu$ occur in different conjuncts "$\forall E2*$" in order to refer to this rule later in defining $\exists I3$. $\forall E2*$ is an equivalence transformation.

universal quantifier elimination 3 ($\forall E3$): A $\bigvee\bigwedge cs\ \mathcal{A}$ implies a $\bigvee\bigwedge cs\ \mathcal{B}$, if $\mathcal{B}$ is generated from $\mathcal{A}$ by eliminating a universal quantifier $\forall\nu$ that is in the scope of an existential quantifier $\exists\mu$ and replacing ν in all its occurrences by μ:

$$\exists\mu\ldots\forall\nu A(\mu,\nu) \vdash \exists\mu A(\nu/\mu)$$

The resulting expression must be converted to a $\bigvee\bigwedge cs$ to achieve $\mathcal{B}$. If necessary, the application of $\forall E3$ must be prepared by PN1-8 to ensure that $\forall\nu$ is in the scope of $\exists\mu$.
It suffices to limit the application of $\forall E3$ to the two following cases:

- $\forall\nu$ is in the scope of $\exists\mu$ and directly succeeds $\exists\mu$, which is the only element of a sequence of existential quantifiers in $\mathcal{A}$. If necessary, $\forall Ex$ and PN1-4 must be applied to satisfy this condition (cf. footnote 20).
- $\forall\nu$ is not in the scope of $\exists\mu$, but $\forall\nu$ and $\exists\mu$ are parts of two expressions that are connected by conjunction. In this case, it suffices to apply the PN-rule to prepare the application of $\forall E3$ (cf. below p. 313).

existential quantifier introduction 1 ($\exists I1$): A $\bigvee\bigwedge cs\ \mathcal{A}$ implies a $\bigvee\bigwedge cs\ \mathcal{B}$, if $\mathcal{B}$ is generated from $\mathcal{A}$ by replacing an arbitrary number i ($0 < i \leq j$) of the j occurrences of a name t with a new variable, μ, bound by a new existential quantifier $\exists\mu$:

$$A(t) \vdash \exists\mu A(t,t/\mu)$$

The resulting expression must be converted to a $\bigvee\bigwedge cs$ to achieve $\mathcal{B}$.
It suffices to limit the application of the rule to the introduction of $\exists\mu$ preceding the resulting expression.

existential quantifier introduction 2 ($\exists I2$)**:** A $\bigvee\bigwedge cs\ \mathcal{A}$ implies a $\bigvee\bigwedge cs\ \mathcal{B}$, if $\mathcal{B}$ is generated from $\mathcal{A}$ by replacing an arbitrary number i ($0 < i < j$) of the j occurrences of a variable μ by a new variable ν. μ is bound by $\exists\mu$ in $\mathcal{A}$, and ν is bound by a new existential quantifier $\exists\nu$ in the scope of $\exists\mu$:

$$\exists\mu A(\mu) \vdash \exists\mu \ldots \exists\nu A(\mu,\mu/\nu).$$

The resulting expression must be converted to a $\bigvee\bigwedge cs$ to achieve $\mathcal{B}$.
It suffices to introduce the new existential quantifier such that $\exists\nu$ directly follows $\exists\mu$ in the resulting expression:

$$\exists\mu A(\mu) \vdash \exists\mu\exists\nu A(\mu,\mu/\nu).$$

existential quantifier introduction 3 ($\exists I3$)**:** A $\bigvee\bigwedge cs\ \mathcal{A}$ implies a $\bigvee\bigwedge cs\ \mathcal{B}$, if $\mathcal{B}$ is generated from $\mathcal{A}$ by replacing an arbitrary number i ($0 < i < j$) of the j occurrences of a variable μ with a new variable, ν. μ is bound by a universal quantifier, $\forall\mu$, and ν by an existential quantifier, $\exists\nu$, occurring in the scope of $\forall\mu$ in the resulting expression:

$$\forall\mu A(\mu) \vdash \forall\mu \ldots \exists\nu A(\mu,\mu/\nu)$$

The resulting expression must be converted to a $\bigvee\bigwedge cs$ to achieve B. If necessary, the application of $\exists I3$ must be prepared by iterative application of $\forall E2*$. This is necessary to ensure that $\exists I3$ is also applied to the effect that the resulting existential quantifier $\exists\nu$ binds variables of different conjuncts, these variables replacing variables of *different* universal quantifiers. In this case $\forall E2*$ must be applied to the effect that $\forall\mu_1, \ldots \forall\mu_k$ are replaced by one universal quantifier $\forall\mu$ and $\mu_1, \ldots \mu_k$ by μ.[21]

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[21] It might be helpful to illustrate this case by a most simple example. From

$$\forall xFxx \wedge \forall yGyy \tag{6.30}$$

one can derive the following $\bigvee\bigwedge cs$:

$$\forall x\exists y(Fxy \wedge Gxy) \tag{6.31}$$

To do so, (6.30) first has to be converted to

$$\forall x(Fxx \wedge Gxx) \tag{6.32}$$

before applying $\exists I3$.

It suffices to limit the application of $\exists I3$ to $\forall\mu$, which is the head of a sequence of universal quantifiers[22] and to $\exists\nu$ directly following $\forall\mu$ in the resulting expression:

$$\forall\mu A(\mu) \vdash \forall\mu\exists\nu A(\mu,\mu/\nu)$$

The application of $\exists I3$ applies to the following two cases:

- The resulting existential quantifier $\exists\nu$ does not bind variables of different conjuncts. In this case it might be necessary to apply $\forall Ex$ and PN1-4 to satisfy the condition that $\forall\mu$ is the head of a sequence of universal quantifiers (cf. footnote 20).
- The resulting existential quantifier $\exists\nu$ does bind variables of different conjuncts. In this case it suffices to apply $\forall E2*$, which implies applying the PN-rule (cf. below p. 313).

$\wedge$ **elimination (**$\wedge E$**):** A $\bigvee\bigwedge cs\ \mathcal{A}$ implies a $\bigvee\bigwedge cs\ \mathcal{B}$, if $\mathcal{B}$ is generated from $\mathcal{A}$ by eliminating some conjunct of $\mathcal{A}$:

$$\bigwedge A, B \vdash A$$

The resulting expression must be converted to a $\bigvee\bigwedge cs$ to achieve $\mathcal{B}$.
It suffices to limit the application of $\wedge E$ to the conjunction of an arbitrary conjunct A and an $\exists$-formula:

$$\bigwedge A, \exists v_1 \ldots \exists v_n [\neg]\varphi(v_1 \ldots v_n) \vdash A$$

with $n \geq 0$.

$\wedge$ **introduction (**$\wedge I$**):** A $\bigvee\bigwedge cs\ \mathcal{A}$ implies a $\bigvee\bigwedge cs\ \mathcal{B}$, if $\mathcal{B}$ is generated from $\mathcal{A}$ by introducing an identical conjunct:

$$A \vdash \bigwedge A, A$$

The resulting expression must be converted to a $\bigvee\bigwedge cs$ to achieve $\mathcal{B}$. However, by applying RCNF-R.2 (cf. p. 249), one must not apply $\bigwedge A, A \vdash A$.[23]

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[22] Here, we still speak of a sequence of universal quantifiers, if they are separated by $\wedge$ or $\vee$, in the logical hierarchy of the formula.

[23] This is obvious because, otherwise, $\mathcal{A}$ and $\mathcal{B}$ would be identical. On the contrary, due to SUB1 and SUB2, as well as the requirement that different quantifiers bind different variables, one yields different conjuncts after converting the formulae to a $\bigvee\bigwedge cs$ if identical conjuncts are not eliminated before applying SUB1 or SUB2. For example, if $\mathcal{A} = \exists x\forall y Fxy \wedge \exists z Gz \wedge \exists x_2 Hx_2$, an application of $\wedge I1$ might yield $\exists x(\forall y_1 Fxy_1 \wedge \forall y_2 Fxy_2) \wedge \exists z Gz \wedge \exists x_2 Hx_2$.

$\top$ **introduction (**$\top I$**):** A $\bigvee\bigwedge cs$ $\mathcal{A}$ implies a $\bigvee\bigwedge cs$ $\mathcal{B}$ if $\mathcal{B}$ is generated from A by introducing a tautologous conjunct:

$$A \vdash \bigwedge A, \top$$

The resulting expression must be converted to a $\bigvee\bigwedge cs$ to achieve $\mathcal{B}$.
It suffices to limit the application of $\top I$ to $\forall$-tautologies:

$$A \vdash \bigwedge A, \forall v_1 \ldots \forall v_n \bigvee \varphi(v_1 \ldots v_n), \neg\varphi(v_1 \ldots v_n)$$

with $n \geq 0$.

$\vee$ **introduction (**$\vee I$**):** A $\bigvee\bigwedge cs$ $\mathcal{A}$ implies a $\bigvee\bigwedge cs$ $\mathcal{B}$, if $\mathcal{B}$ is generated from $\mathcal{A}$ by introducing some disjunct B in $\mathcal{A}$:

$$A \vdash \bigvee A, B$$

The resulting expression must be converted to a $\bigvee\bigwedge cs$ to achieve $\mathcal{B}$.
It suffices to limit the application of $\vee I$ to the introduction of $\forall$-formulae:

$$A \vdash \bigvee A, \bigwedge_1^k \forall v_{1_1} \ldots \forall v_{1_m}[\neg]\psi_1, \ldots, \forall v_{k_1} \ldots \forall_{k_o}[\neg]\psi_k$$

with m,$o \geq 0$ and $k \geq 1$.

$\vee$ **elimination (**$\vee E$**):** A $\bigvee\bigwedge cs$ $\mathcal{A}$ implies a $\bigvee\bigwedge cs$ $\mathcal{B}$, if $\mathcal{B}$ is generated from $\mathcal{A}$ by eliminating an identical disjunct:

$$\bigvee A, A \vdash A$$

The resulting expression must be converted to a $\bigvee\bigwedge cs$ to achieve $\mathcal{B}$.[24]

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[24] It must be kept in mind that we call disjuncts "identical" if they are identical in the corresponding expression of the ab-notation. Thus non-symbolizing features, such as the ordering of conjuncts in a conjunction, disjuncts in a disjunction, existential quantifiers in a sequence of existential quantifiers, universal quantifiers in a sequence of universal quantifiers, and the specific type of variables must be ignored. Nevertheless, by applying SUB1, SUB2, $\exists Ex$, $\forall Ex$, COM$\wedge$, COM$\vee$, ASS$\wedge$, and ASS$\vee$ these differences can be eliminated anyhow.

$\bot$ elimination ($\bot E$): A $\bigvee\bigwedge cs\ \mathcal{A}$ containing a disjunction of a contradiction and an arbitrary disjunct, A, implies a $\bigvee\bigwedge cs\ \mathcal{B}$ if $\mathcal{B}$ is generated from A by eliminating the contradiction:

$$\bigvee \bot, A \vdash A$$

The resulting expression must be converted to a $\bigvee\bigwedge cs$ to achieve $\mathcal{B}$.
It suffices to limit the application of $\bot E$ to $\exists$-contradictions:

$$\bigvee \exists v_1 \ldots \exists v_n \bigwedge \varphi(v_1 \ldots v_n), \neg\varphi(v_1 \ldots v_n), \mathcal{B} \vdash \mathcal{B}$$

with $n \geq 0$.

Note that, according to the limited application of $\top I$ and $\bot E$, it is not necessary to identify tautologies or contradictions in general to identify implications between $\bigvee\bigwedge cs$. Instead, it suffices to identify formulae of a certain syntactic type, i.e. $\forall$-tautologies and $\exists$-contradictions.

$\forall E2$ (and thus $\exists I3$) and $\forall E3$ refer to the PN-rule. The PN-rule defines a succession of applying PN1-8[25] that – after having applied $\forall E2$, $\forall E3$ or $\exists I3$ – results in sequences of quantifiers with a minimal number of existential quantifiers in the scope of universal quantifiers.

PN-rule (RN-R.): Let A and B be the two conjuncts that $\exists\mu$ and $\forall\nu$[26] or two conjuncts / disjuncts $\forall\mu$ and $\forall\nu$ belong to. One should apply the following rules until $\forall\nu$ is in the scope of $\exists\mu$ and $\forall\nu$ is in the scope of $\forall\mu$:

1. Convert A and B by applying PN1-8 such that they are expressions that are preceded by sequences of quantifiers: one sequence ending

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[25] The PN-rules must be applied from the right to the left, i.e. in the direction that can enlarge the scope of quantifiers.

[26] One must keep in mind that the application of the PN-rule presupposes that $\forall\nu$ and $\exists\mu$ occur in different conjuncts according to case 2 of the limited application of $\forall E3$. Application of $\forall E3$, if $\forall\nu$ and $\exists\mu$ occur in different disjuncts, is of no use in this case. This is because application of $\forall E1$, $\exists I1$ and $\exists\forall Ex$ would yield identical results. This is due to the application of PN10 in $\exists$-R, which is needed to convert the resulting expression to a $\bigvee\bigwedge cs\ \mathcal{B}$. Similar remarks hold for $\forall\mu$ and $\forall\nu$ according to case 2 of the limited application of $\forall E2$: Here, it would suffice to refer to different disjuncts because of the application of PN9 in $\forall$-R. However, as we refer to $\forall E2*$ in $\exists I3$ we must also consider the case in which $\forall\mu$ and $\forall\nu$ occur in different conjuncts.

with $\exists\mu$ ($\forall\mu$) and the other sequence ending with $\forall\nu$ ($\forall\nu$).[27]

2. Apply $\forall Ex$ to place $\forall\nu$ ($\forall\mu$ and $\forall\nu$) to the outermost left of a sequence of universal quantifiers.
3. Apply PN1-8 according to the following rules until $\forall\nu$ is in the scope of $\exists\mu$ ($\forall\nu$ is in the scope of $\forall\mu$):
   (a) If PN5-8 is applicable, apply one of these rules.
   (b) If PN1 as well as PN2 are applicable, and if $\forall\nu$ ($\forall\nu$) precedes the left conjunct, then apply PN1. If PN1 as well as PN2 are applicable, and if $\forall\nu$ ($\forall\nu$) precedes the right conjunct, then apply PN2.

This rule guarantees that as few existential quantifiers are in the scope of universal quantifiers as possible after having applied $\forall E2$, $\forall E3$ or $\exists I3$. However, it does not guarantee a unique ordering of existential and universal quantifiers after applying $\forall E2$, $\forall E3$ or $\exists I3$. As long as this only concerns the relative ordering of existential quantifiers in a sequence of existential quantifiers and universal quantifiers in a sequence of universal quantifiers, variations in the ordering are insignificant; these kinds of orderings are not symbolizing properties. Yet, the relative order of existential and universal quantifiers may also vary. In the following

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[27] This does not rule out some avoidable occurrences of existential quantifiers in the scope of universal quantifiers. Thus, we do not rule out that $\exists z$ may occur in the scope of $\forall y$ while applying PN-R.1 in the following example:

$$\forall x(\forall yFxy \vee \forall x_2\exists zGx_2xz) \wedge \forall x_1Hx_1. \tag{6.33}$$

If $\forall\nu = \forall x_1$ and $\exists\mu = \exists z$ application of the PN-R.1 may result in

$$\forall x\forall y\forall x_2\exists z(Fxy \vee Gx_2xz) \wedge \forall x_1Hx_1. \tag{6.34}$$

One could avoid this by laying down further rules of application of PN1-8 in PN-R. 1. However, based on the definition of $\forall E3$ and the application of PN1-8, by generating $\bigvee\bigwedge cs$ and thus minimizing the scope of the quantifiers again, $\exists z$ will not remain in the scope of $\forall y$ after applying $\forall E3$. With respect to the resulting $\bigvee\bigwedge cs$, only the ordering of those quantifiers that have $\exists\mu$ or $\forall\nu$ in their scope in $\mathcal{A}$ are of relevance. Thus, concerning the result of $\forall E3$, it is irrelevant to define a specific application of PN1-8 in PN-R.1. However, for the sake of simplicity, PN1-8 should be applied such that besides the scope of $\exists\mu$ and $\forall\nu$ only the scope of those quantifiers that have $\exists\mu$ and $\forall\nu$ in their scope are changed. In the case of (6.33), one would yield, as a result of PN-R.1:

$$\forall x\forall x_2\exists z(\forall yFxy \vee Gx_2xz) \wedge \forall x_1Hx_1. \tag{6.35}$$

Similar remarks hold in case of preparation of $\exists I3$ by the PN-rule.

example, application of the PN-rule leads to two possible results with $\forall x_4$ in the scope of $\exists y_1$, even after the application of $\forall E3$:

$\bigvee\bigwedge cs\ \mathcal{A}$:

$$\forall x_1\exists y_1\forall x_2 Fx_1y_1x_2 \wedge \exists y_2\forall x_3\exists y_3\forall x_4 Gy_2x_3y_3x_4 \tag{6.36}$$

Application of PN-rule:

$$\exists y_2\forall x_1\exists y_1\forall x_3\exists y_3(\forall x_2 Fx_1y_1x_2 \wedge \forall x_4 Gy_2x_3y_3x_4) \tag{6.37}$$
$$\exists y_2\forall x_3\exists y_3\forall x_1\exists y_1(\forall x_2 Fx_1y_1x_2 \wedge \forall x_4 Gy_2x_3y_3x_4) \tag{6.38}$$

Application of $\forall E3$:

$$\exists y_2\forall x_1\exists y_1(\forall x_2 Fx_1y_1x_2 \wedge \forall x_3\exists y_3 Gy_2x_3y_3y_1) \tag{6.39}$$
$$\exists y_2\forall x_3\exists y_3\forall x_1\exists y_1(\forall x_2 Fx_1y_1x_2 \wedge Gy_2x_3y_3y_1) \tag{6.40}$$

In (6.39), $\forall x_3$ is in the scope of $\exists y_1$, whereas in (6.40), $\exists y_1$ is in the scope of $\forall x_3$. This makes it impossible to apply $\forall E3$ once more causing x_3 to be replaced with y_1 in (6.40). On the other hand, $\forall x_1$ is in the scope of $\exists y_3$ only in (6.40), while in (6.39), $\exists y_3$ is in the scope of $\forall x_1$. This, in turn, makes it impossible to apply $\forall E3$ in (6.39) to cause x_1 to be replaced with y_3. Thus, different possible applications of the PN-rule result in different $\bigvee\bigwedge cs\ \mathcal{B}$ after having applied $\forall E3$. From this, different possibilities of applying $\forall E3$ once more result. Therefore, these different possible applications of the PN-rule are necessary to capture all possible further applications of the rules of implication.

The limited applications of the rules are intended to specify derivations that are "complete" in the sense that no possible applications of other rules depart from those involving $\wedge I$ and $\top I$ are missed out. The limited applications of $\exists\forall Ex$, $\forall E$1-3, and $\exists I$1-3 are all defined such that applications of $\exists\forall Ex$ are not missed out. It is impossible to come to the same results by applying $\exists\forall Ex$ to the antecedent before applying the appropriate rule in its limited way, or by applying $\exists\forall Ex$ to the consequent after having applied the rules according to their limited application (cf. also the explanations on p. 173). Without the limitation that $\exists\mu$ and $\forall\nu$ must be parts of two conjuncts in $\forall E3$, case 2, one can come up with the same results by applying $\forall E1$, $\exists I$, and $\exists\forall Ex$ (cf. footnote 26). The same holds in case of $\exists I3$ according to occurrence of different universal quantifiers in different conjuncts. In addition, applying $\forall E2*$ in $\exists I3$ ensures that no possible applications of $\forall E1$-3 and $\exists I2$ are missed out. $\top I$ can be limited to $\forall$-tautologies, because all other tautologies can be derived from $\forall$-tautologies by applying the rules of implication. In this sense, $\forall$-tautologies constitute the "strongest possible tautologies". Likewise, $\bot E$ can be limited to $\exists$-contradictions because all contradictions contained in any $\bigvee\bigwedge cs$ can be reduced to $\exists$-contradictions, by applying the rules of implication. Contradictions of this form constitute the "weakest possible contradictions".

Contrary to $\wedge I$, $\bot I$, $\vee E$, and $\bot E$, the two rules $\wedge E$ and $\vee I$ do not formulate equivalence rules. They are not valid for the direction from right to left. In order to justify transitions to weaker formulae, $\wedge E$ and $\vee I$ cannot be limited to equivalence rules. If one claimed that $\wedge E$ = $\bigwedge A, \top \vdash A$ and $\vee I$ = $A \vdash \bigvee A, \bot$, then it would be impossible, for example, to justify eliminating conjuncts in other cases. To derive P from $P \wedge Q$, one would first have to derive some tautology from Q. However, as long as one is only allowed to introduce contradictions by $\vee I$, and as long as one is not allowed to eliminate non-tautologous conjuncts, one is unable to derive tautologies from an arbitrary formula. Thus, for the sake of completeness, $\wedge E$ and $\vee I$ are not articulated as rules of equivalence. However, any $\bigvee\bigwedge cs$ can be derived by applying the rules of implication starting with appropriate $\forall$-formula. Furthermore, from any $\bigvee\bigwedge cs$, it is possible to extract a conjunct in the form of an $\exists$-formula, by applying $\forall E1$, $\exists I1$-3. That is why the application of $\vee I$ and $\wedge E$ can be limited to these kind of formulae without limiting the range of implied $\bigvee\bigwedge cs$.

As the general applications of the rules comprise their limited applications, it is of no principal significance to limit their application. To generate the totality of implications between $\bigvee\bigwedge cs$ any possible applications are to be realized anyhow. However, this is necessary if one purports to order $\bigvee\bigwedge cs$ systematically by minimal variations that justify relations of implications. The 13 defined rules of implication satisfy Wittgenstein's criteria of logical operations. They all merely depend on syntactic features of the formulae in question. Obviously, they are merely generalization of the 13 rules of implication defined in section 5.3 for elementary poles.

Similar to the systematic construction of implications between elementary pole-groups the implications between $\bigvee\bigwedge cs$ can be generated iteratively starting from $\forall$-formulae:

starting rule $\bigvee\bigwedge cs$ (S-R. $\bigvee\bigwedge cs$): $\forall$-formulae are the initial formulae of the applications of the 13 rules of implication.

general construction rule (GC-R.): Starting from $\forall$-formulae, all possible applications of the 13 implication rules are to be carried out iteratively if the conditions of their application are satisfied.

The conditions of the application of the rules are defined by their antecedent. They all depend on nothing but syntactic features of $\bigvee\bigwedge cs$ that are varied by applying the rules. Most importantly, they do not depend on the existence of some other relationship of implication. The possible applications of the 13 rules are basically those already specified in case of elementary predicate logic (p. 176, p. 198). However, in case of $\forall E2$ and $\forall E3$ not only quantifier sequences of one closed structure have to be considered. Instead, also the quantifier sequences of any two closed structures to which the rules are applied to must be taken into account. In case of $\forall E2$ (and $\forall E2*$ in $\exists I3$) and $\forall E3$ all possible applications of the PN-rule depending on different possible orders of applying PN-laws have to be considered. The rules of conjunction and disjunction must be applied to any conjunct and disjunct in general. That is not only to conjuncts and disjuncts of the $\bigvee\bigwedge cs$ but also to the conjuncts and disjuncts of the single closed structures. $\top I$ now simply allows to add $\forall$-tautologies by conjunction. Adding closed structures to different disjuncts as in case of $\top$ I in elementary logic is only a special case of this rule that might involve applying further rules. Considering these modifications, the exact number of the possible applications of the 13 rules to $\bigvee\bigwedge cs$ result from slight modifications to their definitions on p. 176 and p. 198.

6.4.2 Correctness and Completeness

6.4.2.1 Correctness

All rules are reducible to applications of known derivation rules. Thus, their correctness follows syntactically. They even apply in the general case of NNFs. Furthermore, the correctness of all the 13 rules can also be proven by paraphrasing the significance of the varying structural property for the truth conditions of the respective $\bigvee\bigwedge cs$. This proof is essentially the same as the proof for the corresponding rules given in elementary predicate logic.[28]

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[28] Some supplementary phrases are necessary to do justice to the fact that the rules may also apply to cases in which one quantifier is not in the scope of the other in the $\bigvee\bigwedge cs$ $\mathcal{A}$, to which

6.4.2.2 Completeness

As in elementary predicate logic we base our completeness proof on the examination of *minimal syntactic variations*. In particular, we prove the two following propositions:

Completeness of GC-R. construction (Prop. 1): By GC-R. the totality of all possible derivations between $\bigvee\bigwedge cs$ according to the syntactic variations specified by our 13 rules of implication is generated.

Completeness of valid minimal syntactic variations (Prop. 2): If some $\bigvee\bigwedge cs$ $\mathcal{B}$ is not derivable from some $\bigvee\bigwedge cs$ $\mathcal{A}$ within the totality of derivations realized by GC-R. any syntactic transition from $\mathcal{A}$ to $\mathcal{B}$ depends on a syntactic variation that constitutes an invalid transition under this condition.

Prop. 1 is not trivially satisfied by the definition of GC-R. It might well be that some $\bigvee\bigwedge cs$ $\mathcal{B}$ would be derivable by the syntactic variations specified by the 13 rules of implication if one would exceed the realm of $\bigvee\bigwedge cs$. That is, it must be shown that converting *wff* to $\bigvee\bigwedge cs$ ensures that the 13 rules of implication are applied to a maximal extent. According to GC-R. all possible applications of the 13 rules of implication must be carried out whenever the conditions of their application are satisfied. To prove Prop. 1 it must also be shown that the conditions of their application are satisfied to a maximal extent. The possibility of such a completeness proof relies on the fundamental idea of New Logic: *To reduce all internal relations of* wff *to systematic variations of syntactic properties by use of equivalence transformations*. If one once comes to understand this fundamental idea to its full extent, comprising its motives and its consequences as well as its difference to the proof conception and meta-logical foundation of traditional logic, it is merely a technical question how to realize it.

Prop. 2 presumes Prop. 1: Given some $\bigvee\bigwedge cs$ $\mathcal{B}$ is not derivable from some $\bigvee\bigwedge cs$ $\mathcal{A}$ according to the totality of possible derivations realized by GC-R., $\mathcal{A}$

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the rules are applied. For example, the semantic proof of the correctness of $\exists\forall Ex$ consists of the following paraphrase:

$\exists\forall Ex$: If *some* object at argument places μ combined with / and *all* objects at argument places ν satisfies / satisfy (does / do not satisfy) $\Im(\varphi)$, then *all* objects at argument places ν combined with / and *some* object at argument places μ satisfy (do not satisfy) $\Im(\varphi)$.

$\Im(\varphi)$ refers to all interpretations of the predicates in which μ and ν occur. The difference between "combined with" and "and", by paraphrasing the truth conditions of the $\bigvee\bigwedge cs$ or pole-groups, is explained in section 6.3.3.

and $\mathcal{B}$ must differ by some syntactic property by virtue of which $\mathcal{B}$ is not implied by $\mathcal{A}$.

Before we prove Prop. 1 and 2, we must clarify the notion of minimal syntactic variations. Strictly speaking, minimal syntactic variations are to be specified between *wff*, more precisely between NNFs in disjunctive normal form, and not between $\bigvee\bigwedge cs$. In this respect, molecular predicate logic differs from elementary predicate logic. In the latter system there is no difference between NNFs in disjunctive normal form and $\bigvee\bigwedge ecs$. In consequence, the rules of implication are definable and applicable without exceeding the realm of $\bigvee\bigwedge ecs$ and the respective pole-groups. In contrast, in molecular logic we must convert NNFs to $\bigvee\bigwedge cs$ before applying the rules of implication. In consequence, it might be necessary to exceed the realm of $\bigvee\bigwedge cs$ to carry out the syntactic variation specified by the respective rule. This is the case, for example, if the PN-rule must be applied before applying $\forall E3$. From this it follows, that it might be impossible to generate a $\bigvee\bigwedge cs$ $\mathcal{B}$ that differs from another $\bigvee\bigwedge cs$ $\mathcal{A}$ in only one syntactic feature. For example, consider $\mathcal{A}$ to be

$$\forall x\forall yFxy \land \exists zGz. \tag{6.41}$$

Applying $\forall E3$ to eliminate $\forall y$ and replacing all occurrences of y by the existential variable z involves applying the PN-rule first. This results in a variation of the orders of quantifiers:

$$\exists z\forall y(\forall xFxy \land Gz). \tag{6.42}$$

This formula is no $\bigvee\bigwedge cs$. Elimination of the universal quantifier then results in the following $\bigvee\bigwedge cs$:

$$\exists z(\forall xFxz \land Gz). \tag{6.43}$$

(6.43) differs from (6.41) not only by the fact that y is replaced by z but also by the fact that $\forall x$ is now in the scope of $\exists z$. It is impossible to generate a $\bigvee\bigwedge cs$ $\mathcal{B}$ that only differs from (6.41) by the elimination of $\forall y$ and the replacement of all the occurrences of y by the existential variable z. Instead, one of the two quantifiers $\forall x$ and $\exists z$ must become part of the scope of the other. However, application of the PN-rule, and more generally, application of rules preparing any specific syntactic variations is merely an equivalence transformation which does not concern symbolizing properties of the formulae. The process of converting *wff* to $\bigvee\bigwedge cs$

is one essential part of ensuring that the conditions of applying the 13 rules of implication are satisfied to a maximal extent. Consider, for example, the *wff*

$$\forall x \forall y \exists z (Fxy \wedge Gz), \tag{6.44}$$

which is equivalent to (6.41). Contrary to the $\bigvee\bigwedge cs$ (6.41), $\forall E3$, or more precisely some general rule of quantifier elimination allowing for replacing the eliminated universal variable by some existential variable of the formula, cannot be applied to the *wff* (6.44) because $\exists z$ occurs in the scope of $\forall y$. However, this ordering of the quantifiers is no symbolizing property. This is demonstrated by the equivalence transformation that reduces the *wff* (6.44) to the $\bigvee\bigwedge cs$ (6.41).

Strictly speaking, only (6.42) differs minimally from (6.43). In general, only the kernel parts of the 13 rules of implication (those which are represented by their formal expressions) specify minimal syntactic variations. However, as we are only concerned with transitions between $\bigvee\bigwedge cs$ we will also say that two $\bigvee\bigwedge cs$ differ *minimally* if they contain *unspecific* syntactic differences in addition to a *specific* minimal syntactic difference. Unspecific syntactic differences are merely due to equivalence transformations. The two $\bigvee\bigwedge cs$ (6.41) and (6.43) differ minimally in the realm of $\bigvee\bigwedge cs$: They differ by the specific variation that y is replaced by z in eliminating $\forall y$. However, this necessarily implies a difference in the scopes of the remaining quantifiers. There is no way in the realm of $\bigvee\bigwedge cs$ for a representation of the former, specific syntactic difference without the latter, unspecific difference. Thus, before actually varying some specific syntactic feature or after having varied some specific syntactic feature it might be necessary to carry out some other syntactic variations by equivalence transformations. However, these unspecific variations are only necessary to prepare the specific syntactic variation or to achieve a $\bigvee\bigwedge cs$ after having varied some specific syntactic feature. They do not constitute some specific variation of a syntactic feature but are only expedient variations of the notation from $\bigvee\bigwedge cs$ to NNF and back to $\bigvee\bigwedge cs$.

On the other hand, there are minimal syntactic variations between NNFs that cannot be retained in the realm of $\bigvee\bigwedge cs$ as they are eliminated by the process of converting NNFs to $\bigvee\bigwedge cs$. For example, the two NNFs

$$\exists x (Fx \vee Gx) \tag{6.45}$$

and

$$\exists x_1 \exists x_2 (Fx_1 \vee Gx_2) \tag{6.46}$$

differ minimally as identical occurrences of an existential variable are replaced with different existential variables. However, this difference cannot be represented in the realm of $\bigvee\bigwedge cs$ because (6.45) and (6.46) are both converted to the following equivalent $\bigvee\bigwedge cs$

$$\exists x_1 Fx_1 \vee \exists x_2 Gx_2. \tag{6.47}$$

Thus, in the realm of $\bigvee\bigwedge cs$ there is no minimal syntactic difference in this case. The syntax of $\bigvee\bigwedge cs$ does not allow for the representation of this non-symbolizing syntactic difference. As no different $\bigvee\bigwedge cs$ correspond to the syntactic difference of the number of existential variables above disjunctions in the realm of NNFs the possibility of derivations between different $\bigvee\bigwedge cs$ is not concerned with this syntactic difference.

According to our conception of New Logic internal relations between *wff*s are reduced to implications between $\bigvee\bigwedge cs$. As the completeness of our calculus for identifying implications between $\bigvee\bigwedge cs$ is at stake, we refer to specific minimal syntactic variations in the realm of $\bigvee\bigwedge cs$. All we claim is that whenever a $\bigvee\bigwedge cs$ $\mathcal{A}$ implies another $\bigvee\bigwedge cs$ $\mathcal{B}$, there is a sequence of $\bigvee\bigwedge cs$ varying as minimal as $\bigvee\bigwedge cs$ can vary syntactically, one implying the other. Thus, we have to consider the validity of specific minimal syntactic differences as represented in the realm of $\bigvee\bigwedge cs$. In question is whether such a sequence is specified by our 13 rules of implication in any case.

In the following, we first prove Prop. 2. We do this by listing all possible specific minimal syntactic variations between $\bigvee\bigwedge cs$ and by demonstrating that only those corresponding to a rule of implication constitute valid transitions. Thus, if the conditions of our 13 rules of implication are satisfied to a maximal extent and it is impossible to derive a $\bigvee\bigwedge cs$ $\mathcal{B}$ from a $\bigvee\bigwedge cs$ $\mathcal{A}$ by GC-R. then $\mathcal{B}$ is not implied by $\mathcal{A}$ because in this case the validity of the implication depends on a a syntactic property that does not constitute a valid transition.

The specific minimal syntactic variations between $\bigvee\bigwedge cs$ form a complete system. Due to the syntax of $\bigvee\bigwedge cs$, and NNFs in general, minimal syntactic variations can be of two kinds: (i) variations concerning argument positions by replacing universal variables, names and existential variables for each other, and (ii) variations concerning introduction and elimination of conjuncts / disjuncts. Both kinds of variations form a closed system. We label the system stemming from variations of sort (i) the "prefix system" and the system stemming from variations of sort (ii) the "suffix system". Two $\bigvee\bigwedge cs$ $\mathcal{A}$ and $\mathcal{B}$ may also vary by one *predicate* in the sense that $\mathcal{B}$ is generated from $\mathcal{A}$ by replacing some predicate (or propositional variable) by some other. However, for the sake of simplicity, we subsume this to eliminating and introducing conjuncts or disjuncts. Likewise, we abstain from considering eliminations or introductions of $\neg$ as the negator only occurs directly left to propositional functions. We also subsume cases of replacing $\wedge$ by $\vee$ or v.v. to elimination and introduction of conjuncts or disjuncts. Thus, tables 6.10 to 6.17 consider all possible combinations of minimal variations concerning universal variables, names, existential variables on the one hand and conjunctions and disjunctions on the other.

Concerning variations of argument positions by replacing universal variables, names and existential variables for each other the following parameters must be considered:

- *all* or only *some* occurrences are replaced.
- occurrences are replaced by a name or a variable not occurring in the formula or by a name or a variable already occurring in the formula. For brevity sake, we label names and variables of the former sort "new names" and "new variables". In contrast, we label names and variables already occurring in the formula "old names" and "old variables".
- in case of universal / existential variables it must be considered which quantifier occurs in the scope of the other.

From this, tables 6.10 to 6.16 arise.

Concerning sentential connectives, the introduction and elimination of $\wedge$ and $\vee$ must be considered. In case of introducing conjuncts it must be distinguished between introducing identical conjuncts, tautologies or arbitrary conjuncts. Likewise one has to distinguish between eliminating identical disjuncts, contradictions and arbitrary disjuncts in case of eliminating disjuncts. Table 6.17 combines these parameters.

The conventions of these tables are explained by the following remarks.

EXPLANATIONS:

- If some minimal syntactic variation does not constitute a relation of implication, we write "–", otherwise we specify the respective rule of implication.
- In tables 6.10 to 6.15, we refer to the fields some/new, some/old, all/new, all/old as fields 1 to 4. Field 1 of table 6.10, for example, refers to replacing *some* occurrences of a universal variable by *new* variables bound by a universal quantifier not occurring in the initial formula.
- In table 6.10, field 3 and table 6.15, field 3 replacing all variables with variables of a new quantifier does not result in different $\bigvee\bigwedge cs$ – the respective

pole remains the same in this case; the substitutions of the variables does not constitute a symbolizing variation. Therefore, these cases do not count for minimal syntactic variations. If we would replace the quantifier by a new quantifier in a different position we simple change the orders of quantifiers. Thus, we abstain from these cases because quantifier exchanges are referred to in table 6.16.

- We subdivide field 4 of table 6.10 and fields 1 and 4 of table 6.12 because the validity of the syntactic variation depends on the relative position of the quantifiers. "$\forall\nu\forall\mu$" in field 4 of table 6.10, for example, means that $\forall\mu$ is in the scope of $\forall\nu$ and not both quantifiers are part of the same sequence of universal quantifiers in $\mathcal{A}$. "$\overline{\forall\nu\forall\mu}$" means that this is not the case. That is, either $\forall\nu$ is in the scope of $\forall\mu$ in $\mathcal{A}$ or none of the two quantifiers is in the scope of the other or they are both part of the same universal quantifier sequence (and thus their relative order can be exchanged by PN1-4 and $\forall Ex$). Similar remarks are valid for $\forall\mu\exists\nu$ and $\overline{\forall\mu\exists\mu}$ in table 6.12, field 1 and field 4. In table 6.12, field 1 this refers to the resulting $\bigvee\bigwedge cs\ \mathcal{B}$, whereas in table 6.12, field 4 we refer to the quantifier order of the initial $\bigvee\bigwedge cs\ \mathcal{A}$.
- Replacing a name or an existential variable by a universal variable and replacing an existential variable by a name or a name by another name does not constitute a relation of implication. We depict all these variations by one and the same table 6.13.
- To complete the prefix system, we have to consider the relative position of quantifiers. As the relation between quantifiers in a sequence of identical quantifiers is no symbolizing syntactic feature, it suffices to refer to the relative position of existential and universal quantifiers in table 6.16.

Table 10: $\forall\mu \Rightarrow \forall\nu$

	new	old	
some	–	–	
all	[SUB 2]	$\forall\nu\forall\mu$ –	$\overline{\forall\nu\forall\mu}$ $\forall E2$

Table 11: $\forall\mu \Rightarrow t$

	new	old
some	–	–
all	$\forall E1$	$\forall E1$

Table 12: $\forall\mu \Rightarrow \exists\nu$

	new		old	
some	$\forall\mu\exists\nu$ $\exists I3$	$\overline{\forall\mu\exists\nu}$ –	–	
all	$\forall E1, \exists I1$		$\forall\mu\exists\nu$ –	$\overline{\forall\mu\exists\nu}$ $\forall E3$

Table 13: $t \Rightarrow \forall\mu$, $t \Rightarrow s$, $\exists\mu \Rightarrow t$, $\exists\mu \Rightarrow \forall\nu$

	new	old
some	–	–
all	–	–

Table 14: $t \Rightarrow \exists\mu$

	new	old
some	$\exists I1$	–
all	$\exists I1$	–

Table 15: $\exists\mu \Rightarrow \exists\nu$

	new	old
some	$\exists I2$	–
all	[SUB 1]	–

Table 16: $\exists\mu\forall\nu \Leftrightarrow \forall\nu\exists\mu$

$\exists\mu\forall\nu \Rightarrow \forall\nu\exists\mu$	$\forall\nu\exists\mu \Rightarrow \exists\mu\forall\nu$
$\exists\forall Ex$	–

Table 17: $\wedge$ I / E and $\vee$ I / E

	$\wedge$			$\vee$		
Introduction	$\bigwedge A, A$ $\wedge I$	$\bigwedge A, \top$ $\top I$	$\bigwedge A, B$ –	$\vee I$		
Elimination	$\wedge E$			$\bigvee A, A$ $\vee E$	$\bigvee A, \bot$ $\bot E$	$\bigvee A, B$ –

Paraphrasing the cases of specific minimal syntactic variations not covered by the 13 rules of implication suffices to recognize that they constitute an invalid transition if the transition from $\mathcal{A}$ to $\mathcal{B}$ depends on one of them (cf. section 6.3.3). In each case the respective syntactic variation identifies a difference in the structure of interpretations that constitutes counter-models. This can be seen by the method described in section 6.3.5. The 13 rules of implication are the only rules that cover minimal syntactic variations that are valid without any further restriction. As rules of implication in terms of *operations* must be valid due to the specific syntactic variations and nothing else there are no other rules of implications in terms of *operations* than these 13 rules. They are the only rules that identify valid transitions in terms of specific minimal syntactic variations if the truth condition of the $\bigvee\bigwedge cs$ rely on the varied syntactic property. As any derivation must not depend on invalid transitions there must be some derivation according to these 13 rules of implication in addition to the use of equivalence transformations, which do not concern symbolizing properties, if some *wff* B is derivable from some *wff* A. Thus, it remains to show that by converting *wff* to $\bigvee\bigwedge cs$ and by referring to GC-R. we indeed realize any possible derivation. This means to show that the 13 rules of implication are applied to a maximal extent.

The question whether this is the case must be distinguished from the question whether the limited applications of the 13 rules do not decrease the construction of implied $\bigvee\bigwedge cs$, which we already answered to the positive (cf. p. 315). Limiting the applications of the rules only discards sequences of applications if one application of a rule comes to the same result. This does neither increase nor decrease the construction of the totality of implied $\bigvee\bigwedge cs$. Limiting the applications of the 13 rules only discards possible applications of the 13 rules that do not increase the derivations of implied $\bigvee\bigwedge cs$. Moreover, by GC-R. we may simply carry out all possible general applications, which necessarily comprise all limited applications. Thus, we do not need to ensure whether no possible derivation is missed by limiting the applications of the rules. In the following, we go on to prove Prop. 1 by showing that GC-R. and our use of equivalence transformations ensure that the 13 rules of implication are applied to a maximal extent.

By the general construction rule GC-R.we presume that all possible applications of our 13 rules are carried out. From this it follows that applications of our rules depending on some other applications of our rules are realized. For example, we cannot apply $\forall E3$ to $\forall x \exists y(Fxy \wedge Gy)$ because $\exists y$ is in the scope of $\forall x$. However, $\forall E3$ is applicable if we first apply $\exists I2$ to derive $\forall x \exists y_1 Fxy_1 \wedge \exists y_2 Gy_2$. From this $\bigvee \bigwedge cs$ we can derive $\exists y_2(\exists y_1 Fy_2 y_1 \wedge Gy_2)$ by applying $\forall E3$. Thus, the maximum of possible applications of the 13 rules depending on other specific syntactic variations is ensured by GC-R. What is to show is the following: Whenever the syntactic conditions of the limited applications of the 13 rules of implication are satisfiable *by equivalence transformation*, these conditions must be satisfied by *our* equivalence transformations. In the following, we first specify the conditions that must be satisfied to ensure a maximum of application possibilities for the respective rules of implication, then we specify the equivalence transformations and show that they ensure the mentioned conditions. Finally, we illustrate this by some examples.

The following syntactic conditions must be satisfied if possible to ensure the application of the 13 rules to a maximal extent (we notice rules depending on the respective syntactic condition in brackets):

C1: Whenever possible existential quantifiers must not occur in the scope of universal quantifiers ($\exists\forall Ex$, $\forall E2$, $\forall E3$).

C2: As many different universal variables as possible must occur ($\forall E1$-3, $\exists I3$). These universal variables are bound by different universal quantifiers.

C3: As many occurrences as possible of any name must occur ($\exists I1$).

C4: As many identical existential variables as possible must occur ($\exists I2$). These existential variables are bound by the same existential quantifier, which is not placed above disjunctions.

C5: As many conjuncts as possible must occur ($\wedge E$).

C6: As many identical and contradictory disjuncts as possible must occur ($\vee E$, $\bot E$).

In contrast to the other rules of implication $\wedge I$, $\top I$ and $\vee I$ do not rely on a special syntactic condition. Thus, there is no need for any equivalence transformation to guarantee their application.

We presume that *wff* s are NNFs in the sense explained on p. 226 and p. 247. Thus, we abstain from any equivalence transformation to obtain NNFs from arbitrary predicate formula by use DN, DM$\vee$, DM$\wedge$, AE, Def. $\neg\forall$, Def. $\neg\exists$, SUB1 and SUB2. In addition, we abstain from ASS$\wedge$, ASS$\vee$, COM$\wedge$ and COM$\vee$ as these rules only concern non-symbolizing orderings of conjuncts and disjuncts. Likewise, we abstain from applications of $\forall Ex$, PN1-4 to vary non-symbolizing orders in sequences of universal quantifiers and we abstain from applications of $\exists Ex$, PN4-8 to vary non-symbolizing orders in sequences of existential quantifiers (cf. footnote 20). All these equivalence transformations are trivial and made superfluous by an adequate notation. Finally, we abstain from equivalence transformations according to RCNF-R. and RDNF-R. involving DM$\wedge$, DM$\vee$, DIS1 and DIS2. RCNF-R. and RDNF-R. are part of $\bigvee\bigwedge cs$-R., they are explained on p. 249. On the one hand, these rules serve to obtain unique propositional normal forms, on the other hand, they prepare the application of PN-laws in order to minimize the scopes of quantifiers as far as possible. The latter function we subsume to the use of PN-laws in order to minimize scopes of quantifiers to a maximal extent. Thus, basically, our equivalence transformations rely on the following two kinds of laws:

1. PN-laws,
2. the 4 equivalence rules $\wedge I$, $\top I$, $\vee E$, $\bot E$.

Converting *wff* to $\bigvee\bigwedge cs$ essentially rests on applying PN-laws. The syntax of $\bigvee\bigwedge cs$ does not allow for formulae in which the scopes of quantifiers are not minimized to a maximal extent. Thus, the syntax of $\bigvee\bigwedge cs$ ensures that certain syntactic conditions of the 13 rules of implications are satisfied if possible. By applying PN-laws in converting *wff* to $\bigvee\bigwedge cs$ formulae not satisfying the syntactic conditions of the 13 rules of implication are reduced to formula satisfying the syntactic conditions. On the other hand, the 4 rules of equivalence serve for the same purpose within the realm of $\bigvee\bigwedge cs$.

Our application of the PN-laws and the 4 equivalence rules satisfy the above mentioned conditions whenever possible for the following reasons:

ad C1: By the use of PN-laws in the process of converting *wff* to $\bigvee\bigwedge cs$ scopes of quantifiers are minimized to a maximal extent. Thus, existential quantifiers do not occur in the scope of universal quantifiers if possible.

ad C2: By use of PN9 in the process of converting *wff* to $\bigvee\bigwedge cs$ the number of universal quantifiers are also increased if possible. Furthermore, by use of $\wedge I$ and $\top I$ universal variables bound by universal quantifiers can be increased to any arbitrary number. By $\wedge I$ one can multiply[29] any universal quantifier $\forall\mu$ occurring in a $\bigvee\bigwedge cs$ by multiplying conjuncts in which μ occurs. By $\top I$ one can introduce any further universal quantifiers of any tautologous condition to any arbitrary extent.

ad C3: Use of $\wedge I$ and $\top I$ before applying $\forall E1$ ensures that any arbitrary number of occurrences of any name can be introduced by $\forall E1$ in addition to their occurrences in a $\bigvee\bigwedge cs$. Furthermore, by $\wedge I$ and $\top I + \forall E1$ occurrences of names can be increased to any arbitrary extent.[30]

ad C4: Use of $\forall E2*$ in preparing $\exists I3$ ensures a maximum of identical existential variables introduced by $\exists I3$. Use of $\wedge I$ and $\top I$ to ensure a maximum of occurrences of any name introduced by $\forall E1$ also ensures that a maximum of identical existential variables can be introduced by $\exists I1$. By use of $\wedge I$ and $\top I + \forall E3$ the number of identical existential variables can be increased to any arbitrary number. By $\wedge I$ one can multiply existential variables by multiplying conjuncts which all contain μ. By $\top I$ one can introduce any $\forall$-tautology as further conjunct to a conjunct containing $\exists\mu$. Applying $\forall E3$ to eliminate a universal quantifier of the $\forall$-tautology and replacing its variable by the existential variable μ increases the occurrences of μ. As the derivation stems from a tautology the resulting $\bigvee\bigwedge cs$ is still equivalent with the initial one. In particular, one has to keep in mind that applying PN10 in the process of converting *wff* to $\bigvee\bigwedge cs$ does not constrain the application of $\exists I2$ as the syntax of $\bigvee\bigwedge cs$ does not allow for identical existential quantifiers above disjunctions (cf. p. 320).

ad C5: By $\wedge I$ and $\top I$ as many conjuncts as possible can be introduced by equivalence transformation.

ad C6: C6 is satisfied by the fact that any $\forall$-formula can be introduced as a disjunct by $\vee I$[31] and the fact that the maximum of application possibilities of

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[29] That a universal quantifier $\forall\mu$ is "multiplied" means that all the occurrences of $\forall\mu$ (or $\forall\mu_1 \ldots \forall\mu_k$ after applying SUB2) precede the same expression.

[30] However, although this increases possibilities to apply $\exists I1$, this does not constitute further relations of implication as one can derive the same results by use of $\wedge I$ and $\top I + \forall E3$ to the results of applying $\exists I1$.

[31] We subsume applications of $\forall\bigvee\bigwedge cs$-S. to $\vee I$ here.

> the other rules to this $\forall$-formula is ensured. Deriving identical or contradictory disjuncts under this condition proves that the introduced $\forall$-formula and all the formulae derived from it are redundant. Thus, the application of $\vee I$ plus all the applications of the 13 rules of implication to the introduced disjunct are equivalent transformations.

Thus, Prop. 1 holds.

In the following we illustrate how our equivalence transformations reduce derivations to minimal syntactic variations specified by our 13 rules of implication.

By the process of converting *wff* to $\bigvee\bigwedge cs$ it is revealed when the relative order of existential and universal quantifiers is no symbolizing property. Thus, for example, instances of table 6.10, field 4(a) can be converted to instances of table 6.10, field 4(b). This is already illustrated by (6.44) (= $\forall x\forall y\exists z(Fxy \wedge Gz)$) and (6.41)(= $\forall x\forall yFxy \wedge \exists zGz$) on p. 319. C1 also applies to $\forall E2$ because $\overline{\forall\nu\forall\mu}$ (cf. table 6.10, field 4) is satisfied whenever possible if C1 is satisfied whenever possible. Consider, for example, the following *wff*

$$\forall x\exists y\forall z(Fx \vee Fyz) \tag{6.48}$$

$\forall E2$ cannot be applied to (6.48) such that z is replaced by y. However, this is possible in case of the equivalent $\bigvee\bigwedge cs$:

$$\forall xFx \vee \exists y\forall zFyz \tag{6.49}$$

From (6.49)

$$\exists y\forall z(Fz \vee Fyz) \tag{6.50}$$

can be derived by applying $\forall E2$ whereas this is not possible in case of (6.48).

Converting *wff* to $\bigvee\bigwedge cs$ also increases the number of universal quantifiers. This is significant for applying $\forall E1$-3 and $\exists I3$. For example, the following minimal variation is no instance of the 13 rules of implication:

$$\forall x(Fx \wedge Gx) \wedge \exists yHy \vdash \exists y(Fy \wedge Hy) \wedge \forall xGx \tag{6.51}$$

Instead, it is an instance of table 6.12, field 2. However, by converting the premise to the equivalent $\bigvee\bigwedge cs$ the relation of implication is reduced to following instance of $\forall E3$:

$$\forall x_1Fx_2 \wedge \forall x_2Gx_2 \wedge \exists yHy \vdash \exists y(Fy \wedge Hy) \wedge \forall x_2Gx_2. \tag{6.52}$$

Similar considerations are valid regarding the possibility to apply $\forall E1$. Applicability of $\forall E2$ and $\exists I3$ are not affected by increasing universal quantifiers due to PN9 in the process of converting *wff* to $\bigvee\bigwedge cs$. This is because $\forall E2$ cannot result in different formulae in this case because converting the result of the quantifier elimination back to a $\bigvee\bigwedge cs$ countermands the quantifier elimination. $\exists I3$ on the other hand is not affected by increasing universal quantifiers due to converting *wff* to $\bigvee\bigwedge cs$ because this rule allows for replacing only some universal variables by existential variables.

However, $\forall E2$ and $\exists I3$ as well as $\forall E1$ and $\forall E3$ are all affected by multiplying universal quantifiers due to $\wedge I$. This allows for realizing different possible applications of these rules. The following examples demonstrate this:

$$\exists y(\forall x Fxy \wedge Gy) \vdash \exists y(Fay \wedge Fby \wedge Gy) \tag{6.53}$$

$$\forall x\forall y\forall z Fxyz \vdash \forall x\forall z Fxxz \wedge \forall x\forall y Fxyx \tag{6.54}$$

$$\forall x Fx \wedge \exists y Gy \wedge \exists z Hz \vdash \exists x(Fx \wedge Gy) \wedge \exists z(Fz \wedge Hz) \tag{6.55}$$

$$\forall x Fxxx \vdash \forall x\exists y Fxyy \wedge \forall x\exists z Fzxx \tag{6.56}$$

By first applying $\wedge I$ different applications of $\forall E1 - 3$ and $\exists I3$ are made possible.

The following example is an instance of table 6.15, field 2:

$$\exists x(Fx \wedge \exists y(\neg Fy \wedge \exists z(Gxz \wedge Gzy))) \tag{6.57}$$

$$\vdash$$

$$\exists x(Fx \wedge \exists y(\neg Fy \wedge Gxy \wedge \exists z Gzy)) \tag{6.58}$$

Some occurrence of the existential variable z in (6.57) is replaced by the "old" variable y in (6.58). Such a substitution is not valid in general. However, by use of $\top I$ the derivation of (6.58) from (6.57) is reducible to syntactic transitions specified by our 13 rules of implication:

(1)	$\exists x(Fx \wedge \exists y(\neg Fy \wedge \exists z(Gxz \wedge Gzy)))$	(6.57)
(2)	$\exists x(Fx \wedge \exists y(\neg Fy \wedge \exists z(Gxz \wedge Gzy))) \wedge \forall z_1(Fz_1 \vee \neg Fz_1)$	$\top I$
(3)	$\exists x(Fx \wedge \exists y(\neg Fy \wedge \exists z(Gxz \wedge Gzy \wedge Fz))) \vee$ $\exists x(Fx \wedge \exists y(\neg Fy \wedge \exists z(Gxz \wedge Gzy \wedge \neg Fz)))$	$\forall E3$

(4)	$\exists \mathbf{x}(\mathbf{Fx} \wedge \exists \mathbf{y}(\neg \mathbf{Fy} \wedge \exists z(Gzx \wedge \mathbf{Gxy} \wedge Fz))) \vee$ $\exists \mathbf{x}(\mathbf{Fx} \wedge \exists \mathbf{y}(\neg \mathbf{Fy} \wedge \exists z(\mathbf{Gxy} \wedge Gyz \wedge \neg Fz)))$	see below
(5)	$\exists x(Fx \wedge \exists y(\neg Fy \wedge Gxy)) \vee$ $\exists x(Fx \wedge \exists y(\neg Fy \wedge Gxy))$	$4 \times \wedge E$
(6)	$\exists x(Fx \wedge \exists y(\neg Fy \wedge Gxy))$	$\vee E$
(7)	$\exists x(Fx \wedge \exists y(\neg Fy \wedge Gxy \wedge Gxy))$	$\wedge I$
(8)	$\exists x(Fx \wedge \exists y(\neg Fy \wedge Gxy \wedge \exists z Gzy))$	$\exists I2$

Line (4) is merely a different notation of the same $\bigvee\bigwedge cs$ – the respective pole is identical. We applied SUB1 to replace x with z and z with x in the first disjunct, whereas we replaced z with y and y with z in the second disjunct. In addition, we applied ASS$\wedge$, COM$\wedge$, PN5, PN6 and $\exists Ex$ to change the order of quantifiers and conjuncts. The different notation makes transparent that $\exists x(Fx \wedge \exists y(\neg Fy \wedge Gxy))$ is contained in both disjuncts. In line (5) we united several applications of $\wedge E$. This example also illustrates that relationships of implication are not only *identified* by reducing derivations to the syntactic transitions specified by our 13 rules of implication. Instead, it is also *explained* why some formula is implied by another. (6.57) implies (6.58) because any object at the positions of z in models of the premise is either identical with an object that satisfies $\Im(F)$ or that does not satisfy $\Im(F)$. This becomes transparent by the a derivation based on $\top I$ according to our calculus.

In general, our procedure of reducing implications between *wff* to derivations between $\bigvee\bigwedge cs$ according to our 13 rules of implication *explains* the relationship of implication between predicate formula because they reduce it to those syntactic variations that alone *justify* the relation of implication: By converting a *wff* A and B to $\bigvee\bigwedge cs$ and deriving $\mathcal{B}$ from $\mathcal{A}$ by our rules of implication any syntactic difference between A and B that does not constitute a relation of implication is reduced to syntactic variations that are valid without any restriction. It is in virtue of these syntactic variations why B is implied by A. Such a concept of explaining relations of implication is not available in traditional logic.

To sum up, our calculus is complete because relations of implication between $\bigvee\bigwedge cs$, and in addition implications between *wff* by conversion to $\bigvee\bigwedge cs$, are reduced to syntactic variations specified by our 13 rules of implication and if no such reduction is possible the respective formula imply some syntactic difference that discards a relation of implication.

As our 13 rules of implication are operations and as the totality of all relations of implications is defined by the iterative application of the 13 rules due to GC-R. the problem of implication is solved.

6.4.3 Decidability

In section 5.3.7.4 we defined two possible decision procedures based on our 13 rules of implication in the realm of elementary logic: One by defining upper bounds for the application of GC-R. in elementary predicate logic, and one that reduces the question of implication between elementary pole-groups, or $\bigvee\bigwedge ecs$ respectively, to the question of implication between elementary poles, or ecs respectively. The strategy for the latter kind of decision procedure was based upon correlating pole-groups. However, we cannot correlate disjuncts of $\bigvee\bigwedge cs$ likewise because closed structures can itself contain disjunctions. We also cannot reduce the question of implication between $\bigvee\bigwedge cs$ to the question of implication between single cs because all sorts of internal relations between single cs and $\bigvee\bigwedge cs$ exist. Because of the same reasons it is also not obvious how to define upper bounds for the application of $\forall E1$, $\wedge I$, $\top I$ and $\vee I$ in molecular logic. However, this does not mean that no strategies are available that specify decision procedures on the basis of our 13 rules of implication. Nor is Church's theorem a conclusive reason not to look for such procedures within the conception of New Logic because the proof of Church's theorem has no probative force according to New Logic (cf. section 3).

However, we will not discuss the possibility to define a decision procedure for pure first order logic here. Our intention is not to refute modern mathematical logic. Instead, our object is to refute the judgement that Wittgenstein's conception of logic is refuted by modern mathematical logic.

As the solution of the equivalence problem implies the solution of the decision problem, we do not propose a solution for this final problem. However, our solution of the equivalence problems in elementary predicate logic relies on the definition of an unambiguous strategy to minimize pole-groups on the basis of identifying relations of implication according to our 13 rules of implication. This strategy can trivially be brought forward to molecular logic if the decision problem were solvable on the basis of our calculus in molecular logic. Thus, the solution of the equivalence problem does not depend on anything else than the solution of the decision problem.

6.5 Summary of Part II

In Part II we elaborated the conception of New Logic by first realizing the ab-notation for elementary predicate logic and then expanding it to the whole realm of predicate logic.

Elementary predicate logic is the part of predicate logic that does not contain dyadic sentential connectives in the scope of quantifiers. In this part of predicate logic, we were able to realize the conception of New Logic in its entirety. We solved the *equivalence problem* by defining a purely syntactic procedure to convert elementary predicate formulae to ab-symbols. These represent the truth conditions of the initial formulae unambiguously. This solution consists of the following steps:

wff ⇒ ab-diagram ⇒ pole-groups ⇒ symbolizing pole-groups ⇒ ab-symbol

The general guide of this solution is to identify the symbolizing properties of ab-notation, in other words, those syntactic properties that are necessary to identify the truth conditions of *wff*s by syntactic features. The process of reducing symbolizing pole-groups to ab-symbols is basically a generalization of the first step of the Quine-McCluskey algorithm. This generalization, in turn, refers to our solution of the *problem of implication* by defining 13 rules of implication in terms of logical operations. 7 of the 13 rules apply to the quantifiers and names, 6 additional rules to the introduction and elimination of poles. These rules specify those minimal syntactic variations that constitute valid transitions between poles and pole-groups. We showed that the totality of implications in the realm of elementary predicate logic is constructible by the iterative application of these rules. On the basis of these rules we specified a decision procedure for the relation of implication between pole-groups in the realm of elementary predicate logic.

In chapter 6 we discussed how to expand the conception of New Logic to the whole realm of pure predicate logic. First, we explained how to generate pole-groups for arbitrary *wff*s. For this reason, we defined rules of translation between *pole-groups* and *disjunctions of conjunctions of closed structures* $\bigvee \bigwedge cs$, as well as a procedure to convert *wff*s to $\bigvee \bigwedge cs$, solely by the application of known equivalence rules. Contrary to prenex normal forms, the scope of quantifiers is minimized as much as possible according to PN-laws in closed structures. We refer to closed structures, or equivalently predicate poles, as the *unanalysable expressions*, which are the basis of any explanation of truth conditions in predicate logic within New Logic. They are the analogue to negated and non-negated atomic propositions (literals) of propositional logic in predicate logic. On this basis, we defined the concept of *New Semantics*, which solves the *problem of semantics* for the whole realm of predicate logic. The all-important feature of $\bigvee\bigwedge cs$ and, more accurately, of the pole-group notation, is the capability of identifying structural features of models and counter-models of the initial *wff* by syntactic features. For this reason, models and counter-models are not only identified but it is also *explained* by virtue of which structure interpretations are models and counter-models. Furthermore, it is possible to define rules that can generate the totality of models and counter-models of a *wff* on the basis of the syntactic features of its representation using the pole-group notation. However, unlike ab-symbols, unreduced pole-groups still suffer from the deficiency that the same classes of models / counter-models may be represented by different syntactic features. In this respect, we pointed out that a complete realization of the intention of New Semantics, to represent truth conditions of *wff*s unambiguously, presupposes the solution of the equivalence problem. Finally, we solved the *problem of implication* for the whole realm of predicate logic by generalizing the 13 rules of implication already used in elementary predicate logic. We demonstrated that any relation of implication in predicate logic is reducible to an iterative application of these rules. Relations of implications between *wff* are not merely identified but also *explained* by specifying a chain of minimal specific syntactic differences between the corresponding $\bigvee\bigwedge cs$ that all constitute relations of implications. New Logic reduces internal relations of *wff* to systematic variations of syntactic properties by use of equivalence transformations.

However, contrary to elementary predicate logic, we did not define a decision procedure based on our calculus for the whole realm of predicate logic. In consequence, we did not solve the *equivalence problem*. Our intention was to show that New Logic is not less powerful than Old Logic. We did not intend to refute Old Logic. Instead, we wanted to refute the common judgement that Wittgenstein's idea to identify truth conditions of predicate formulae by the syntactic properties of their adequate representation would not be realizable. His conception of logic, comprising its philosophical motivations as well as its conception of a logical proof, is not refuted by modern mathematical logic. Instead, if spelled out, it forms a strong alternative to it that should be considered on the basis of its own criteria.

Thus, our attempt to elaborate New Logic ends up with rivaling paradigms of logic. The reasons to judge in favour of the traditional conception of logic are not conclusive. One must take into account that Wittgenstein's conception deviates significantly from the standards of modern mathematical logic in a reasonable way. Basic notions (p.r. functions, actual infinite), methods (diagonalization, logical formalization of arithmetic and formalizing formal properties by characteristic function, evaluating interpretations $\Im$ one by one), fundamental assumptions (Church's thesis), and theorems (Church's theorem) of the traditional Paradigm, as well as the relevance of its problems (define a correct and complete calculus) and their typical solutions (axiomatic proof conception), are denied on the basis of Wittgenstein's rejection of the possibility to adequately consider formal properties and relations by referring to non-syntactic properties. Instead, on the basis of his way of a purely syntactic foundation of logic other notions (operation, symbolizing property) and other methods (reducing syntactic differences of equivalent formulae, systematic variation of symbolizing properties, generating models and counter-models by iteration) are established. The main object is to *explain* truth conditions of *wff* and their internal relations by syntactic features of their ideal representations. Problems (equivalence problem, problem of semantics, problem of implication) that are neither posed nor solvable within the traditional picture are taken into account. Hopefully, the attraction of this New Paradigm is increased in devising new strategies to solve these problems and in their partial solution. This book has fulfilled its purpose if it has convinced the reader that New Logic is a serious alternative to mathematical logic, and that, even in such a well known topic as pure predicate logic, things are not as certain and unchangeable as they might seem to those who never considered that an alternative may exist after the evolution of modern mathematical logic. Hopefully, this encourages fresh minds to get involved in New Logic and to elaborate similar ideas in the realm of mathematics.

Appendix A

Derivation Rules

PN-Laws

$\forall\nu(A \wedge B(\nu)) \dashv\vdash A \wedge \forall\nu B(\nu)$ PN1
$\forall\nu(B(\nu) \wedge A) \dashv\vdash \forall\nu B(\nu) \wedge A$ PN2
$\forall\nu(A \vee B(\nu)) \dashv\vdash A \vee \forall\nu B(\nu)$ PN3
$\forall\nu(B(\nu) \vee A) \dashv\vdash \forall\nu B(\nu) \vee A$ PN4
$\exists\nu(A \wedge B(\nu)) \dashv\vdash A \wedge \exists\nu B(\nu)$ PN5
$\exists\nu(B(\nu) \wedge A) \dashv\vdash \exists\nu B(\nu) \wedge A$ PN6
$\exists\nu(A \vee B(\nu)) \dashv\vdash A \vee \exists\nu B(\nu)$ PN7
$\exists\nu(B(\nu) \vee A) \dashv\vdash \exists\nu B(\nu) \vee A$ PN8
$\forall\nu(A(\nu) \wedge B(\nu)) \dashv\vdash \forall\nu A(\nu) \wedge \forall\nu B(\nu)$ PN9
$\exists\nu(A(\nu) \vee B(\nu)) \dashv\vdash \exists\nu A(\nu) \vee \exists\nu B(\nu)$ PN10

Further Equivalence Rules

$A \dashv\vdash A$ A
$\neg\neg A \dashv\vdash A$ DN
$(A \vee B) \wedge C \dashv\vdash (A \wedge C) \vee (B \wedge C)$ DIS1
$(A \wedge B) \vee C \dashv\vdash (A \vee C) \wedge (B \vee C)$ DIS2

$\neg(A \vee B)$	$\dashv\vdash$	$\neg A \wedge \neg B$	DM$\vee$
$\neg(A \wedge B)$	$\dashv\vdash$	$\neg A \vee \neg B$	DM$\wedge$
$A \rightarrow B$	$\dashv\vdash$	$\neg A \vee B$	AE
$A \wedge A$	$\dashv\vdash$	A	IP1
$A \vee A$	$\dashv\vdash$	A	IP2
$(A \vee \neg A \vee B) \wedge C$	$\dashv\vdash$	C	IP3
$(A \wedge \neg A \wedge B) \vee C$	$\dashv\vdash$	C	IP4
$(A \vee B) \wedge (A \vee \neg B)$	$\dashv\vdash$	A	IP5
$(A \wedge B) \vee (A \wedge \neg B)$	$\dashv\vdash$	A	IP6
$A \wedge B$	$\dashv\vdash$	$B \wedge A$	COM$\wedge$
$A \vee B$	$\dashv\vdash$	$B \vee A$	COM$\vee$
$(A \wedge B) \wedge C$	$\dashv\vdash$	$A \wedge (B \wedge C)$	ASS$\wedge$
$(A \vee B) \vee C$	$\dashv\vdash$	$A \vee (B \vee C)$	ASS$\vee$
$\exists \mu A(\mu)$	$\dashv\vdash$	$\exists \nu A(\nu)$	SUB 1
$\forall \mu A(\mu)$	$\dashv\vdash$	$\forall \nu A(\nu)$	SUB 2
$\neg\forall \mu A(\mu)$	$\dashv\vdash$	$\exists \mu \neg A(\mu)$	Def. $\neg\forall$
$\neg\exists \mu A(\mu)$	$\dashv\vdash$	$\forall \mu \neg A(\mu)$	Def. $\neg\exists$
$\forall \mu \forall \nu A(\mu,\nu)$	$\dashv\vdash$	$\forall \nu \forall \mu A(\mu,\nu)$	$\forall Ex.$
$\exists \mu \exists \nu A(\mu,\nu)$	$\dashv\vdash$	$\exists \nu \exists \mu A(\mu,\nu)$	$\exists Ex.$

Abbreviations

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MS 107: manuscript 107 according to von Wright's catalogue. Published in *Wittgenstein's Nachlass: the Bergen Electronic Edition*, Oxford University Press: London 2000 and in WA2 (see below).

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PG: Wittgenstein, Ludwig: *Philosophical Grammar*, Blackwell: London 1974.

PM: Russell, Bertrand und Whitehead, Alfred North: *Principia Mathematica*, Cambridge: University Press, Vol. I: 1910, Vol II: 1912, Vol III: 1913, 1. Edition.

PR: *Philosophical Remarks*, Blackwell: Oxford, 1975.

RFM: *Remarks on the Foundations of Mathematics*, Blackwell: Oxford 1956.

TLP: Wittgenstein, Ludwig: *Tractatus Logico-Philosophicus*, Routledge: London 1994.

WA2: Wittgenstein, Ludwig: *Wiener Ausgabe, Band 2*, Springer: Wien, New York 1994.

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